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HIGH-VELOCITY HYDRODYNAMICS, NO. 4

I. L. Rozovskiy, Editor

"Naukova Dumka" Press, Kiev, 1968

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Table of Contents

NON-CLASSICAL PROBLEMS AND METHODS IN PERTURBATION THEORY[1]

A. N. Panchenkov

ABSTRACT. The author considers the problem of "complete" approximation of a boundary value problem in some region of the Euclidian of a new boundary value problem. The criteria for a uniform approximation are satisfied by allowing freedom in the selection of the mapping of the Euclidian space into the new space as well as in the choice of the new differential operator, the boundaries of the region and the boundary conditions. The complete approximation in the new space yields a linear problem which leads to satisfactory results in physical space for sufficiently large perturbation parameters. The Poincare-Lighthill-Go method is an example of a procedure based on the idea of complete approximation. Problems in hydrodynamics which may be solved by this method are examined.

The most universally used method for solving boundary value problems in mathematical physics with small perturbations is the method of constructing a formal solution as a series in positive hours of a small parameter of the problem. The first approximation of this method which yields a solution with an accuracy to terms of the first negative order of magnitude ordinarily corresponds to some linear problem, and an iteration process of the formal algorithm for a method in perturbation theory gives results of higher approximations. /9*

In evaluating the advantages of methods in perturbation theory, two problems must be studied: 1) the actual possibility of finding the results of higher approximations; 2) the theoretical possibility of finding the results of higher approximations.

A series of positive powers of a small parameter gives a formal solution for the problem, but there is a sharp increase in the difficulty of constructing the inverse operator with a transition from the first to higher approximations. For this reason only first approximation theories are completely worked out in most cases (for instance the theory of small amplitude waves in a heavy liquid).

Cases are described in the literature where formal solutions of higher approximation have not improved on the accuracy of the first approximation. In this case we are running up against problems in which it was fundamentally impossible to improve on the results of the first approximation by an ordinary method in perturbation theory. Let us formulate conditions which when satisfied

[1]Report given at the World Congress of Mathematicians in Moscow in 1966.

*Numbers in margin indicate pagination in foreign text.

will open up the fundamental possibility of realizing a formal algorithm of a method of perturbation theory, and examine the problem of "complete" approximation of a boundary value problem by a new boundary value problem.

Let Ω and Ω_0 be regions of Euclidian space R^3 whose boundaries belong to the class l_2: /10

$$B \in l_2; \qquad B_0 \in l_2.$$

Let us designate by class l_m regions bounded by a finite number of simple closed curves intersection in pairs [1].

Each curve which makes up part of the boundary of region Ω_i has a finite length, and the function $x_i = x_i(s)$ which determines this curve has continuous derivatives to order m inclusive (s is the arc length reckoned from a fixed point on the curve).

Let us consider two sets in the space $C^2(\Omega_i)$:

$$\Phi \in C^2(\Omega);$$
$$\Phi_0 \in C^2(\Omega_0).$$

The sets Φ_i satisfy differential equations of the second order

$$T\Phi = 0 \qquad g \in \Omega;$$
$$T_0\Phi_0 = 0 \qquad g \in \Omega_0,$$

and the boundary conditions

$$K\Phi = 0 \qquad g \in B;$$
$$K_0\Phi_0 = 0 \qquad g \in B_0.$$

on the boundaries $B_i \in l_2$.

Let P be the mapping $\Omega \to \Omega_{01}$, and $B_{01} \in l_2$ be the boundary of the region Ω_{01}. Let us introduce in spaces $C^2(\Omega_0 \cap \Omega_{01})$ and R^3 the metrics

$$\varrho_c(P\Phi\Phi_0); \quad \varrho_R(B_0 : B_{01}) = \sqrt{\sum_{i=1}^{3}(x_1^i - x_2^i)^2}; \quad x_1^i \in B_0; \qquad x_2^i \in B_{01} \qquad (i = 1, 2, 3),$$

where the points $g(x_1^i)$ and $g(x_2^i)$ are the intersections of surfaces B_0 and B_{01} respectively by the normal to the surface B_{01}.

Let us assume that the function Φ_0 gives a uniform approximation in the region Ω_0:

$$P\Phi = \Phi_0 + 0(\varepsilon^\gamma) \qquad g \in \Omega_0,$$

where

$$\gamma > 0; \qquad \varepsilon < 1;$$
$$\|\Phi_0\| \sim 1.$$

If the conditions /11

$$\varrho_c(P\Phi;\ \Phi_0) \approx 0(\varepsilon^\alpha) \qquad \alpha > 0;$$
$$\varrho_R(B_0 : B_{01}) \approx 0(\varepsilon^\beta) \qquad \beta > 0,$$

are satisfied which are necessary for finding solutions of the problem by perturbations methods, the question of the approximation with the greatest exponent γ becomes important.

In classical perturbation theory, the mapping $P \equiv 1$ and linear theories as a first approximation in methods of perturbation theory give an approximation with an exponent $\gamma = 1$ in most cases.

Complete approximation means that with free selection of the mapping P, an attempt may be made to select the mapping P and the operators T_0 and K_0 in such a way that they guarantee the greatest exponent γ.

If the perturbation algorithm increases both exponents α and β, there will also be an increase in exponent γ, whereas if the algorithm increases only one exponent, the results of higher approximations will yield the same accuracy as the first approximation. In this regard, it is necessary to construct an algorithm of perturbation theory which increases exponents α and β; such an algorithm may be constructed by introducing the mapping $P \not\equiv 1$. Closest to the idea of complete approximation is the Poincare-Lighthill-Go method [8].

Let us examine three problems of aerohydrodynamics in which the method of complete approximation has been used to find some interesting results in the first approximation [5, 6].

Non-linear Theory of a Supporting Surface or Arbitrary Extension

Linear theories have been developed in considerable detail in this problem, but there are no strict solutions of the non-linear problem.

Let us assume that the flow to be considered is Baltrami flow. Since it is clear from the physical essence of the problem to be considered that the greatest effect will be from the presence of the downwash angle at infinity behind the wing and the deviation of the vortex sheet from the plane OXZ, let us examine the mapping with deformation of the coordinate Z alone.

To satisfy the second condition let us take the relationship for the coordinate in the form

$$Z_1 = Z + \int_0^{x_1} v_{z_1}(\tau, y_1 : Z_1)\, d\tau.$$

For acceleration potential, the boundary value problem is formulated in the following way:

$$\begin{gathered} \Delta\Theta = 0 \qquad g \in \Omega_0; \\ -\frac{1}{v_0}\int_{\infty}^{x_1} \Theta_{z_1} d\tau = F(g) \qquad g \in S_0; \\ \Theta \to 0 \\ x \to +\infty \\ \Theta_+ - \Theta_- = 0 \qquad g \in L_1. \end{gathered} \tag{1}$$

/12

Here L_1 is the trailing edge of the supporting surface and the last condition is the condition of the Zhukovskiy-Chaplygin postulate.

Boundary value problem (1) in the new space is equivalent to the linear boundary value problem for the supporting surface [7]. By solving problem (1) by ordinary methods in space R^3, we get a two dimensional integral equation

$$\varphi_{Z_1} = \frac{1}{4\pi}\int_{S_0} \frac{\gamma(p)}{\lambda(\eta)} \cdot \frac{\partial}{\partial y}\left\{\frac{(y-\eta)}{(y-\eta)^2 + \left(\frac{F(\xi)}{\lambda(\eta)}\right)^2} \times \right.$$

$$\left. \times \left[1 - \frac{\lambda^2(\eta)(y-\eta)^2 + (x-\xi)^2 + 2F^2(\xi)}{(x-\xi)\sqrt{\lambda^2(\eta)(y-\eta)^2 + (x-\xi)^2 + F^2(\xi)}}\right] ds\right\} \qquad g \in S.$$

where $\lambda(\eta)$ is the relative longitudinal extension in cross-section η.

To approximate a wing of small extension we have a one-dimensional integral equation

$$\frac{1}{2\pi\lambda}\int_{-1}^{+1}\frac{\Phi_\eta\,(y-\eta)}{(y-\eta)^2+\left(\frac{2\beta(\eta)}{\lambda}\right)}\,d\eta=\sin\alpha \qquad g\in S_0,$$

whose solution gives the value of the lift factor for a wing of small extension

$$C_y=\frac{\pi\lambda}{2\psi_3}\sin\alpha,$$

where

$$\psi_3=2\int_0^\infty e^{-\frac{2\beta}{\lambda}k}\,\frac{I_1^2(k)}{K}\,dK.$$

The function ψ_3 is computed as

$$\psi_3=0,5\tau_\beta^2+0,25\tau_\beta^4+0,0625\tau_\beta^6+0,0169\tau_\beta^8+0,0237\tau_\beta^{10}+$$
$$+0,0188\tau_\beta^{12}+0,0981\tau_\beta^{14}.$$

here /13

$$\tau_\beta=\sqrt{\left(\frac{\beta}{2\gamma}\right)^2+1}-\frac{\beta}{2\lambda}; \qquad \beta=\frac{a\sqrt{\lambda}}{2\sqrt{2}},$$

where a = 1 for a wing with a rounded edge; a = 2 for a wing with a sharp trailing edge.

The results of this solution agree satisfactorily with experimental data on wings of small extension (Fig. 1).

Supporting Surface in a Near Sonic Gas Flow

The problem of motion of a body in a subsonic gas flow [6] was considered as a problem in which the operator T varies.

The motion of the fluid is described by the function $\phi\in C^2(\Omega)$ which satisfies the nonlinear equation of gas dynamics [2]:

$$[1-M^2-\varepsilon M^2(\gamma+1)\varphi_x]\,\varphi_{xx}+\varphi_{yy}+\varphi_{zz}+0(\varepsilon^2)=0, \quad \text{where } ||\phi||\sim 1.$$

The mapping P^{-1} is considered with components: /14

$$P^{-1} = \begin{cases} x = \frac{\xi}{k} + F(\xi, \eta, \zeta); \\ y = \eta; \\ z = \zeta; \end{cases}$$

$$F(\xi, \eta, \zeta) \in C^2(\Omega) \qquad \|F\| \sim \varepsilon.$$

In the new space R_1^3 in the first approximation we get $\Delta\phi = 0$, $g \in \Omega_0$, and for the function F we get the equation

$$[(1 - M^2) K^2 - 1] \varphi_{\xi\xi} - \varepsilon 2K^2 F_{\xi} (1 - M^2) \varphi_{\xi\xi} -$$
$$- \varepsilon K^2 F_{\xi\xi} (1 - M^2) \varphi_{\xi} - \varepsilon M^2 (\gamma + 1) \varphi_{\xi\xi} \varphi_{\xi} K^3 + 0 (\varepsilon^2) = 0 \qquad g \in \Omega_0$$

If we take the function F in the form

$$F = B(\varphi - \xi C) \qquad g \in \Omega_0,$$

we have for the constants K and B

$$k^3 \varepsilon \frac{2}{3} C M^2 (\gamma + 1) - K^2 (1 - M^2) + 1 = 0;$$
$$B = - \frac{M^2 (\gamma + 1)}{3(1 - M^2)}.$$

To define the constant C we assume the condition

$$F_{\xi}(-k, 0) = 0.$$

Summing up these results we find that the problem of gas dynamics is formulated in space R_1^3 for the Laplace equation, and the coordinate transformation $\xi \to x$ takes the form

$$x = \frac{\xi}{K} - \varepsilon \frac{M^2 (\gamma + 1)}{3(1 - M^2)} [\varphi - \xi \varphi_{\xi} (-k, \eta)].$$

For small perturbations this expression becomes a well-known transformation which leads to the Prandtl-Glauert dimensionless number.

We shall not dwell on the methods for solving the corresponding boundary value problems, but shall merely give a number of final results which show the effectiveness of the method.

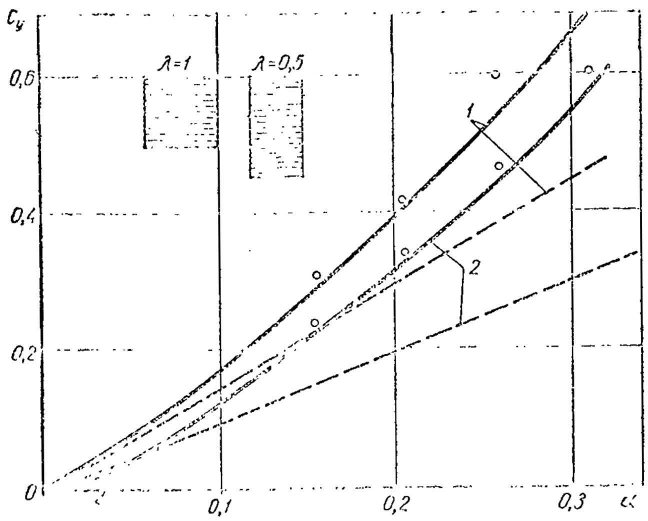

Figure 1. Comparison of Theoretical and Experimental Data on Wings of Small Extension (the broken line is computed from linear theory): 1, λ =1; 2, λ = 0.5.

According to the theory under consideration, the pressure coefficient is

$$C_p = \frac{2\varepsilon\varphi_\xi K}{1 - \frac{\varepsilon K M^2(\gamma+1)}{3(1-M^2)}\left[\varphi_\xi - \varphi_\xi(-k, 0)\right]}.$$

Special cases:

1. For a symmetric elliptical foil in a plane parallel flow, we get the dimensionless equation /15

$$K^3 \frac{\delta}{3} M^2(\gamma+1) - K^2(1-M^2) + 1 = 0,$$

where δ is the relative thickness of the wing.

It follows from this equation that in the region

$$\frac{\delta}{3} M^2 \ll (1-M^2)$$

the Prandtl-Glauert number applies, while in the region

$$\frac{\delta}{3} M^2 \sim (1-M^2)$$

we get the near sonic dimensionless parameter

$$K = \frac{\lambda}{\delta^{1/3} M^{2/3} (\gamma+1)^{1/3}}.$$

from the equation.

2. For a thin wing in plane flow, we find that the general dimensionless number is the Prandtl-Glauert number, while the pressure coefficient should be calculated from the formula /16

$$C_p = \frac{C_{p_0}}{\sqrt{1-M^2}\left(1 + \frac{M^2(\gamma+1)C_{p_0}}{6(1-M^2)^{3/2}}\right)}.$$

to account for non-linear effects, where C_{p0} is the pressure coefficient in an incompressible liquid.

A comparison of the results of computation by the formula with known solutions in gas dynamics [3] shows that they agree satisfactorily even for perturbations of the order of 1 (Fig. 2).

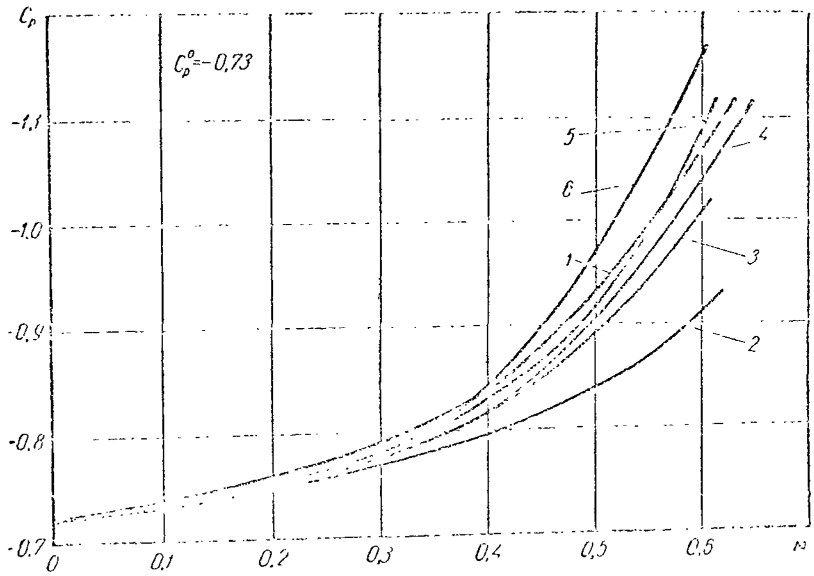

Figure 2. Comparison of Various Methods for Computing the Pressure Coefficient from an Incompressible Liquid to a Gas:
1, Stack experiment (profile NACA4412, upper surface $\alpha = 0°15'$); 2, by the Prandtl-Glauert formula; 3, by the Karman-Chiang formula; 4,5, by G. A. Dombrovskiy's formula; 6, by formula (32).

The problem of a supporting surface in a 3-dimensional subsonic flow was also considered, and a nonlinear 2-dimensional integral equation was found as well as final results for the pressure field on the wing surface.

The Problem of Waved Finite Amplitude

This problem on the surface of a heavy liquid is formulated as follows (Fig. 3).

$$\Delta\varphi = 0 \qquad g \in \Omega^1;$$
$$\omega_1\eta_b - \varphi_x + (\nabla\varphi)^2 = 0;$$
$$\varphi_x\eta_{bx} - \eta_{bx} = \varphi_y \qquad g \in L_b$$
$$\nabla\varphi \to 0$$
$$x \to \infty.$$

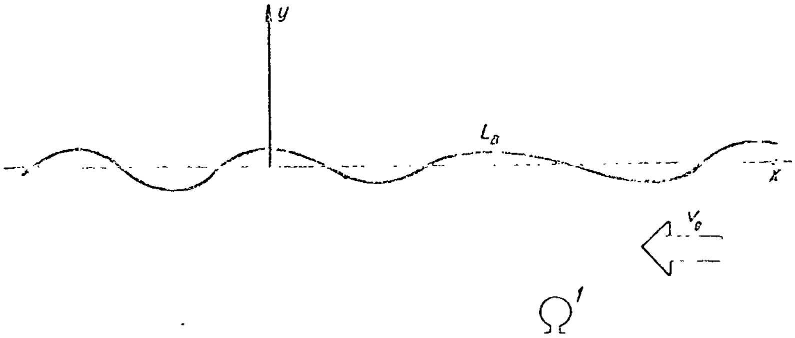

Figure 3. Coordinate System for Solving the Problem of Waves on the Surface of a Liquid.

Let us assume that

$$\eta_0 \in C^m(L_0); \qquad \varphi \in C^n(\Omega') \qquad (n \geq 2)$$

and

$$\|\varphi\| \sim \|\eta_0\|; \qquad \|\eta_0\| < 1.$$

Let us limit ourselves to the approximation

$$F(g) = F_0(g_1) + \eta_0 F_{0\eta}(g_1) + 0(\varepsilon^2) \qquad g \in L_0;$$
$$F(g) \in C^n(\Omega'); \quad F_0(g) \in C^n(\Omega); \quad \|F_0(g)\| \approx \varepsilon \quad g_1 \in L.$$

Then the nonlinear problem for moderate perturbations takes the form

$$\Delta\varphi = 0 \qquad g \in \Omega;$$
$$\varphi_{xx} + \omega_1\varphi_y - \frac{\partial}{\partial x}(\nabla\varphi)^2 - \frac{\partial}{\partial x}\varphi_x^2 + 0(\varepsilon^3) = 0 \qquad g \in L;$$

$$\Delta\varphi \to 0;$$
$$x \to +0.$$ /17

Let us introduce a new Euclidian space R_1^2. For the sake of simplicity we again consider a space with a single distorted coordinate

$$x = \xi + F(\xi, \eta); \qquad y = \eta.$$

In space R_1^2, the problem takes the following form:

$$\Delta\varphi = 0 \qquad g \in \Omega_0;$$
$$\varphi_{\xi\xi} + \omega_1\varphi_\eta = 0 \qquad g \in L$$

with the mapping $\xi \to x$;

$$x = \xi - \varphi(\xi, \eta).$$

The mapping $\xi \to x$ is taken from the condition that the first approximation in space R_1^2 should correspond to the classical theory of low amplitude waves.

The shape of the profile in space R_1^2 is determined from the formula

$$\eta_b = \frac{1}{\omega_1}\varphi \qquad g \in L.$$

For the space R^2, we have the parametric representation:

$$x = \varepsilon - \varphi(\xi, 0);$$
$$\eta_b = \frac{1}{\omega_1}\varphi_\xi(\xi, 0).$$

In the simplest case of waves with a potential in space R_1^2

$$\varphi = Ae^{\omega_1\eta}\sin\omega_1\xi,$$

while in the space R^2 /18

$$x = \xi - A\sin\omega_1\xi;$$
$$\eta_b = A\cos\omega_1\xi. \tag{2}$$

Formulas (2) describe trochoidal Gerstner waves. Thus we find that the first approximation in space R_1^2 gives the known theory of trochoidal Gerstner waves [4].

Only the simplest example from wave theory has been examined, but the given method may be used for studying an extremely broad class of nonlinear problems which are not likely to be solvable by conventional methods.

Let us point out in conclusion that the methods of perturbation theory which are based on complete approximation give first-approximation nonlinear solutions which are suitable for fairly large perturbations.

There is a large class of problems in mathematical physics which may be studied by the given method.

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A SUPPORTING SURFACE IN AN UNSTEADY FLUID FLOW
(Article 1)

A. N. Panchenkov

ABSTRACT. The author presents a general method of solving boundary value problems for small hydrodynamic perturbations of a supporting surface in an unsteady flow. The method is based on representation of the general solution in the form of three components. The first two components give the regular solutions which describe inertial and vortex motions respectively, while the third gives the singular solution. The problem of oscillation of a wing in an infinite fluid is given by way of example.

The theory of a wing in an unsteady flow has been quite extensively /19
developed [1-4, 7, 8]. Work which has recently been done in high velocity hydrodynamics and hydrobionics shows that the theory of a wing in an unsteady flow demands extensive investigations to develop general methods and new mathematical models which could be used for solving a broad class of problems on a supporting surface (rigid and elastic) in a bounded flow, problems on unsteady motions of a wing with finite perturbations, developing a theory for the thrust of a flapping wing, etc.

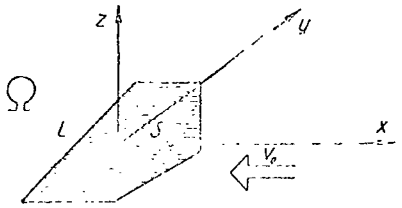
The introduction of high speed water vehicles has been favorable to the development of a theory of unsteady motion of a wing in a bounded flow (close to a free surface, close to a solid screen), but it has been found that the available data in aerodynamics are insufficient for such generalizations. Even in the case of plane flow with a closed solution for the Birnbaum equation for a wing close to a deflector, new methods must be worked out for approximate solution of the more general equation.

The method of acceleration potential may be used for solving the problem of a supporting surface in an unsteady flow. This method leads to 2-dimensional singular equations which are extremely difficult to solve for a supporting surface of arbritrary shape and arbritrary conditions.

Works [1-4] which take up this theory deal chiefly with the problem of various physical assumptions and approximations which reduce the 2-dimensional equations to 1-dimensional equations. However, as is pointed out by R. L. Bissplinghoff, H. Ashley and R. L. Halfmann [1], the problem has been handled /20
in about 20 different ways and no 2 treatments give completely identical results.

Beginning with this paper, we intend to publish a series of articles dealing with the general theory of a wing in an unsteady flow and with its applications in high velocity hydrodynamics and hydrobionics.

The approximate original plan for the studies is as follows:



1) formal apparatus of the theory of small perturbations and general results;

2) plane problems in the theory of a wing in an unsteady bounded flow;

3) general solutions in 3-dimensional problems;

4) 1-dimensional approximations in the theory of a wing of finite span;

5) numerical results and applications;

6) methods of "complete" approximation and the theory of moderate perturbations;

7) applications of the theory of small and moderate perturbations in hydrobionics (the theory of flapping flight, thrust of a vibrating wing, etc.).

Works [4-6] contain the basic results necessary for realizing this program the initial data needed for understanding the material which has been published. The mathematical apparatus of the theory of small perturbations is developed in this paper.

The analysis is based on the coordinate system shown in the figure.

Let us introduce the following basic notation: θ, acceleration potential; ϕ, velocity potential; k, Strouhal number; N, the mapping $\theta \rightarrow \phi$; N_0, the mapping $\theta \rightarrow \phi$ for steady-state flow.

Let us assume that in addition to its fundamental translational motion with velocity v_0, the wing undergoes harmonic oscillations with frequency ω. We then have for the hydrodynamic potential and their derivatives

$$\begin{aligned} \theta^n_{x_i}(g, t) &= \bar{\theta}^n_{x_i}(g)\, e^{i\omega t}; \\ \varphi^n_{x_i}(g, t) &= \bar{\varphi}^n_{x_i}(g)\, e^{i\omega t} \end{aligned} \qquad (1)$$

We have used the index n to designate the derivative of n-th order of the functions θ and ϕ with respect to the coordinate x_i. Expressions (1) may be used in formulating the problem for potentials $\bar{\theta}(g)$; $\bar{\Phi}(g)$.

In the following discussion we shall omit the lines above the notation for the amplitude functions and will consider dimensionless values with a velocity of the oncoming flow equal to unity. /21

The relationship between potentials θ and ϕ is

$$\theta = -ik\varphi + \varphi_x.$$

The boundary value problem for a supporting surface in an unsteady flow is formulated as follows

$$\Delta\theta = 0 \qquad g \in \Omega; \tag{2}$$

$$\theta_z = F(g) \qquad g \in s_p; \tag{3}$$

$$\theta \to 0;$$

$$x \to +\infty;$$

$$\begin{aligned} P\theta &= F_1(g) \qquad g \in s_0 \\ \theta_+ - \theta_- &= 0 \qquad g \in L. \end{aligned} \tag{4}$$

Here Ω is the region occupied by the fluid; s_p is the projection of the supporting surface on the surface of the undisturbed flow; s_0 is the bounding surface; L is the trailing edge of surface s_p.

The linear operator P determines the structure of the boundary conditions on the bounding surface. For example:

for a free surface

$$P = \frac{\partial^2}{\partial x^2} + \frac{1}{Fr^2}\frac{\partial}{\partial z};$$

for a solid deflector

$$P = \frac{\partial}{\partial z}.$$

The solution of problem (2) gives an integral operator $A\gamma$ with domain in the space $L_1(s_p)$, range in the space $c^2(\Omega)$, and the properties

$$\Delta A\gamma = 0 \qquad g \in \Omega;$$

$$A\gamma_+ = \bar{A}\gamma + \frac{1}{2}\gamma \qquad g \in s_{p+};$$

$$A\gamma_{-} = \bar{A}\gamma - \frac{1}{2}\gamma \qquad g \in s_{p-};$$

$$A_z\gamma_{+} = A_z\gamma_{-} = \bar{A}_z\gamma \qquad g \in s_p.$$

The additional properties of operator $A\gamma$ are defined by condition (4).

Since

$$A\gamma_{+} - A\gamma_{-} = + \gamma = 0_{+} - 0_{-} \qquad g \in s_p,$$

the norm of the function $\gamma(g)$ in space $L_1(s_p)$ is proportional to the relative /22
lift:

$$\|\gamma\| = \int_{s_p} \gamma(g)\, ds.$$

The integral equation for the problem takes the form

$$\bar{A}_z\gamma = F(g) \qquad g \in s_p. \tag{5}$$

Examples of construction of an integral operator for equation (5) show [4] that the kernel of the integral operator has a singularity of order $\alpha = 2$ and the intetrals do not exist in the general case, but may exist only for $\gamma(g) \in c^1(s_p)$.

Since N^{-1} is a differential operator and formula (5) gives a solution only in the space $c^1(s_p)$, we may expect that there is a singular solution which is lost in the mapping $\phi \to \theta$. The singular solution should be determined from the equation

$$N\bar{A}_z\gamma = F_1(g) \qquad g \in s_p. \tag{6}$$

The proper selection of the integral operators from expression (6) gives all the known equations in the theory of a wing in an unsteady flow [1, 3, 4].

The operators $A(\gamma)$ in the general case may be written in the following form form:

for 2-dimensional flow
$$A\gamma = \frac{1}{2\pi}\int_{-1}^{+1} \gamma(\zeta)\frac{\partial}{\partial\eta} K_1^2(\zeta, g)\, d\zeta \qquad g \in \Omega;$$

for 3-dimensional flow

$$A\gamma = \frac{1}{4\pi} \int_{s_p} \gamma(p) \frac{\partial}{\partial\zeta} K_1^3(p, g)\, ds \qquad g \in \Omega.$$

For a wing in an unbounded liquid, the kernels K_1^2 take the form

$$K_1^2 = \ln \frac{1}{R_2};$$

$$K_1^3 = \frac{1}{R_3},$$

where R_n is the distance in Euclidian space R^n (the upper index associated with the kernel shows the dimensionality of the space).

Further development of the theory is along the lines of working out analytical or numerical methods for solution of equation (6).

With respect to physical essence, the function γ may be written as /23

$$\gamma(p) = \gamma_1(p) + \gamma_2(p) \qquad p \in s_p.$$

where the term $\gamma_1(p)$ describes the vortex components of the solution, and $\gamma_2(p)$ describes the inertial components.

$$\|\gamma_1\| \leqslant M; \qquad \|\gamma_2\| \leqslant kM_1 \tag{7}$$

These estimates follow from the already known results of wing theory [3].

If k ~ 1, we get

$$\|\gamma\| \sim \|\gamma_1\| \sim \|\gamma_2\| \sim M;$$

and numerical methods may then yield satisfactory results.

When k >> 1, we have

$$\|\gamma_1\| \ll \|\gamma_2\| \text{ and } \|\gamma\| \sim \|\gamma_2\|;$$

and numerical methods may then be used for a rough determination of the vortex components which are of greatest interest in the theory of a wing in an unsteady flow.

This fact and the possibility of finding effective analytical solutions determine the importance of the problem of representing the solution in the form of individual components which have either different physical meanings or different norms in the corresponding metric spaces.

Several methods of representing the solution are known in the literature.

In A. I. Nekrasov's monograph [3], the solution is broken down according to physical meaning:

1) quasistationary solution;

2) inertial component of the solution;

3) solution describing the effect of the vortex trail.

In the most recent papers, the solution is broken down according to formal mathematical meaning [1]:

1) regular solution ($c^1(s_p)$);

2) singular solution.

These two methods have certain disadvantages: in the first case the quasistationary solution and the solution which describes the effect of the vortex trail do not belong to the space $c^1(s_p)$, while in the second case, the regular solution has components with different norms.

In the formal algorithm which we have developed, θ is represented in the form of three components:

$$\theta = \theta_1 + \theta_2 + \theta_3, \tag{8}$$

where θ_1 is the regular solution associated with a discontinuity in tangential velocities when a line in s_p is crossed; θ_2 is the regular solution which describes inertial motions; θ_3 is the singular solution of the problem. /24

If the condition

$$\varphi_z = F_1(g) \qquad g \in s_p,$$

is observed in velocity potential space, then

$$F(g) = ikF_1(g) - F_{1x}(g) \qquad g \in s_p \tag{9}$$

and formula (5) resolves into two independent equations:

$$\bar{A}_z\gamma_1 = -F_{1x}(g) \qquad g \in s_p;$$
$$\bar{A}_z\gamma_2 = ikF_1(g) \qquad g \in s_p$$

or

$$N_{01}\bar{A}_z\gamma_1 = F_1 + c \qquad g \in s_p; \tag{10}$$
$$N_{01}\bar{A}_z\gamma_2' = -ikF_1(g) \qquad g \in s_p; \tag{11}$$
$$N_{01} = -\int^{x} d\tau.$$

A necessary condition for the existence of equation (11) is $\gamma_2 \in c^1(s_p)$. Assuming that the functions belong to space $c^1(s_p)$, we shall consider these functions finite.

Expressions (10) and (11) imply the important relationship

$$\gamma_2'(g) = -ik\gamma_1(g) + \frac{\partial}{\partial x}\gamma_{22}(g) \qquad g \in s_p; \tag{12}$$
$$N_0A_z\gamma_{22}' = ikc \qquad g \in s_p. \tag{13}$$

The constant c is determined from the condition of solvability in the space $c^1(s_p)$ which is clearly formulated for 2-dimensional problems, and in the theory of long extension, the value of this constant used in the 2-dimensional problem may be assumed (for 3-dimensional problems, the constant c will be parametrically dependent on the coordinate y). For instance, for the equation

$$L_0\gamma = F_1(s) + c \qquad s \in [-1; +1];$$
$$L_0\gamma = \frac{1}{2\pi}\int_{-1}^{+1}\frac{\gamma(s)}{(x-s)}\,ds$$

is a singular operator with Cauchy kernel.

The constant c is determined from the condition /25

$$c = -\frac{1}{\pi}\int_{-1}^{+1}\frac{F_1(s)}{\sqrt{1-s^2}}\,ds,$$

and an inverse operator L_0^{-1} then exists in this class.

The following may be shown in determing the singular solution:

$$N\bar{A}_z\gamma_{12} = N_{01}\bar{A}_z\gamma_1 \qquad g \in s_p. \tag{14}$$

If

$$\gamma_{12} \in c^1(s_p) \text{ и } \gamma_2(x, y) = -ik\int_{-1}^{x} \gamma_1(\tau, y)\, d\tau; \qquad \{x, y\} \in s_p,$$

then the expression

$$\begin{aligned} N\bar{A}_z\gamma_3 &= LC \qquad g \in s_p, \\ L &= -I - N\bar{A}_z P_1, \end{aligned} \tag{15}$$

follows from equations (6) and (14) for the singular solution, where I is the identity operator;

$$N_0\bar{A}_z P_1\gamma'_{22} = \gamma_{22}.$$

The basic characteristics and advantages of the formal algorithm which has been developed are now apparent. Functions γ_1 and γ_2 are determined from equation (10) and (11), which coincide in form with the equation of stationary theory, while the singular solution γ_3 is determined from equation (15) which contains only the constant c in the right-hand member.

If it is assumed that the Zhukovskiy-Chaplygin postulate is valid throughout the entire period of motion, a singularity will exist only on the leading edge of the wing.

The form of the singular solution is known for 2-dimensional flow $\left(\gamma_3(x) = a\sqrt{\frac{1+x}{1-x}}\right)$, and in this case we immediately get the value of the constant a from expression (15).

To illustrate the method, let us briefly solve the plane problem of oscillation of a wing in an unbounded fluid. For this form of the operator, equations (10), (11) and (13) take the form

$$\left.\begin{aligned} L_0\gamma_1(s) &= F_1(x) + c \\ L_0\gamma_2' &= -ikF_1 \\ L_0\gamma_{22}' &= ikc \end{aligned}\right\} \quad x \in [-1; +1]. \tag{16}$$

Introducing the inverse operator L_0^{-1} power, we have /26

$$\gamma_{22}'(x) = -\frac{2ikcx}{\sqrt{1-x^2}} + \frac{2}{\pi}\frac{D}{\sqrt{1-x^2}} \qquad x \in [-1; +1];$$

$$\gamma_1(x) = \frac{2}{\pi}\sqrt{1-x^2}\int_{-1}^{+1}\frac{F_1}{\sqrt{1-s^2}}\frac{1}{(s-x)}\,dx \qquad x \in [-1; +1]. \tag{17}$$

If the singularity at the leading edge of the wing alone is considered in the singular solution, the general representation for the function $\gamma(s)$ may be written as

$$\gamma(s) = a_0\sqrt{\frac{1+s}{1-s}} + 2ikc\sqrt{1-s^2} + \frac{2}{\pi}D\int_{-1}^{s}\frac{1}{\sqrt{1-x^2}}\,dx +$$

$$+ \gamma_1(s) - ik\int^{s}\gamma_1(x)\,dx \qquad x \in [-1; +1]. \tag{18}$$

Direct inversion of the second equation in system (16) gives

$$\gamma_2'(x) = -\frac{2}{\pi}\frac{ik}{\sqrt{1-x^2}}\int_{-1}^{+1}\frac{\sqrt{1-\tau^2}F_1(\tau)}{(\tau-x)}\,d\tau \qquad x \in [-1; +1];$$

$$\gamma_2(x) = -\frac{ik}{\pi}\int_{-1}^{+1}\ln\left|\frac{(1-px)+\sqrt{1-p^2}\sqrt{1-x^2}}{(1-px)-\sqrt{1-p^2}\sqrt{1-x^2}}\right|F_1(p)\,dp \tag{19}$$

$$x \in [-1; +1].$$

The constant D may be determined from the condition of equality of the norms of the space $L_1(s_p)$ with respect to the two solutions

$$\|\gamma_2(x)\| = -ik\left\|\int_{-1}^{s}\gamma_1(\tau)\,d\tau\right\| + \|\gamma_{22}\|.$$

Then
$$D = ik\int_{-1}^{+1}\frac{F_1 x}{\sqrt{1-x^2}}\,dx.$$

The constant a is determined from equation (15): /27

$$\left.\begin{aligned}
a &= -2\pi\frac{\left(1+\frac{ik}{\pi}N_2\right)c-\frac{c_2}{2\pi}N_3}{N_1};\\
N_1 &= -\pi + ike^{ikx}\int_{-1}^{+1}\sqrt{\frac{1+s}{1-s}}\int_{\infty}^{x}\frac{e^{-ik\tau}}{(\tau-s)}\,ds\,d\tau;\\
N_2 &= \int_{-1}^{+1}\frac{\sqrt{1-s^2}}{(x-s)}\,ds+ike^{ikx}\int_{-1}^{+1}\sqrt{1-s^2}\int_{\infty}^{x}\frac{e^{-ik\tau}}{(\tau-s)}\,ds\,d\tau;\\
N_3 &= ike^{ikx}\int_{-1}^{+1}\frac{1}{\sqrt{1-s^2}}\int_{\infty}^{x}\frac{e^{-ik\tau}}{(\tau-s)}\,ds\,d\tau;
\end{aligned}\right\} \tag{20}$$

$$c_2 = -\frac{2i}{k\pi}D.$$

The function N_i may be expressed in terms of Hankel functions:

$$\left.\begin{aligned}
N_1 &= ik\frac{\pi^2}{2}\left[H_1^{(2)}(k)+iH_0^{(2)}(k)\right]e^{ikx};\\
N_2' &= +\frac{ik}{\pi}N_2 = -ik\frac{\pi}{2}H_1^{(2)}(k)\,e^{ikx};\\
N_3 &= k\frac{\pi^2}{2}H_0^{(2)}(k)\,e^{ikx}.
\end{aligned}\right\} \tag{21}$$

Then we get

$$a = 2c(k)\left[c+\frac{c_2}{2}\right]-c_2, \tag{22}$$

where c(k) is Theodorsen's function [3].

In conclusion, let us give an expression for the norm of the function γ in space $L_1(s)$ which, as has already been pointed out, is proportional to the relative lift:

$$\|\gamma\| = \bar{P} = -2\,[c(k)\int_{-1}^{+1}\sqrt{\frac{1-x}{1+x}}F_1dx+ik\int_{-1}^{+1}\sqrt{1-x^2}F_1dx. \tag{23}$$

Expression (23) implies estimates (7) which were assumed a priori. Expression (23) is a known result from aerodynamics [3] on the lift of an oscillating wing in a 2-dimensional flow.

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PROBLEMS OF LONG WAVES WHICH ARISE FROM INITIAL PERTURBATION IN A VISCOUS ROTATING LIQUID

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ABSTRACT: The author considers the problem of long waves which arise on the surface of a viscous rotating liquid under the effect of an initial elevation in the free surface.

Problems on long waves which arise from initial perturbations in an ideal rotating liquid are considered in [2, 5 and 10] and in a viscous liquid without regard to rotation--in [3, 4, 6 and 9]. /29

In this paper we investigate two-dimensional and three-dimensional problems of long waves which arise on the free surface of a viscous rotating liquid from an arbitrary initial elevation. The analogous problem without regard to rotation is solved in [8].

Let us consider an infinite layer of a viscous liquid bounded from below by a horizontal floor, and from above by the free surface. The liquid is at rest at the initial instant of time, and its free surface, which is displaced from the horizontal position, has the form

$$\zeta(x, y, 0) = af(x) \quad (a = \text{const}) \tag{1}$$

It is necessary to determine the form of the free surface at any instant of time t > 0, assuming that the waves which arise on the surface are long and that they move slowly, and taking the Coriolis force into account. Under these conditions, with initial and boundary conditions

$$\begin{gathered} u = v = 0, \quad \zeta = af(x) \text{ when } t = 0; \\ u = v = 0 \text{ when } z = -h; \; u_z = v_z = 0 \text{ when } z = 0 \end{gathered} \tag{2}$$

the system of equations of motion is as follows [7]:

$$\frac{\partial u}{\partial t} - 2\omega v = -g\frac{\partial \zeta}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}\,;\quad \frac{\partial v}{\partial t} + 2\omega u = \nu\frac{\partial^2 v}{\partial z^2}\,;$$

$$\frac{\partial \zeta}{\partial t} = -\int_{-h}^{0}\frac{\partial u}{\partial x}\,dz. \qquad (3)$$ /30

Taking for the solution of system (3) the Laplace transform with respect to time t and the Fourier transform with respect to the variable x and satisfying conditions (2), we get for the form of the free surface:

$$\zeta(x,\,t) = \frac{a}{2\pi i}\int_{s-i\infty}^{s+i\infty}\frac{\alpha^2+4\omega^2}{\alpha}\,\varphi(\alpha)\,e^{\alpha t}\,d\alpha; \qquad (4)$$

$$\varphi(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(m)\,\frac{1}{\Delta(m,\,\alpha,\,\varepsilon)}\,e^{imx}\,dx;$$

$$f(m) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-imx}\,dx; \qquad (5)$$

$$\Delta(m,\,\alpha,\,\varepsilon) = \alpha^2 + 4\omega^2 + m^2\left[1 - \frac{1}{2\alpha}\,\varepsilon\left(\frac{\alpha-2\omega i}{\sqrt{\alpha+2\omega i}}\,\mathrm{th}\,\frac{\sqrt{\alpha+2\omega i}}{\varepsilon} + \frac{\alpha+2\omega i}{\sqrt{\alpha-2\omega i}}\,\mathrm{th}\,\frac{\sqrt{\alpha-2\omega i}}{\varepsilon}\right)\right].$$

Here x, t and ω are dimensionless quantities equal respectively to $x\sigma c^{-1}$, σt and $\omega\sigma^{-1}$, where $\sigma = 1\ \sec^{-1}$, $c = \sqrt{gh}$, $\varepsilon = \nu^{1/2}\sigma^{1/2}h^{-1}$.

Formula (4) gives a representation for the expression of the form of the free surface for an arbitrary initial elevation which may be represented by a Fourier integral, and for arbitrary values of the parameter ε. Since $f(m) \to 0$ when $|m| \to \infty$, integral (5) will converge, and since $\alpha\phi(\alpha) \to 0$ when $|\alpha| \to \infty$, integral (4) will also converge.

Since the parameter ε is small for real liquids over a wide range of a variation in h, it is of interest to analyze expression (4) further for small values of this parameter.

Expanding the integrand in equation (5) in a series with respect to parameter ε, limiting ourselves to the first and third terms of the series, and writing the residual term in Lagrangian form, we get expression (4) in the form

$$\zeta = \zeta_0 + \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \varepsilon^3\zeta_3; \tag{6}$$

$$\zeta_k = \frac{a}{2\pi i}\int\limits_{s-i\infty}^{s+i\infty} \frac{\alpha^2 + 4\omega^2}{\alpha^{k+1}}\varphi_k(\alpha)\, e^{\alpha t} d\alpha; \tag{7}$$

$$\varphi_k = \frac{1}{2^k\sqrt{2\pi}}\left[\frac{\alpha - 2\omega i}{\sqrt{\alpha + 2\omega i}} + \frac{\alpha + 2\omega i}{\sqrt{\alpha - 2\omega i}}\right]^k \int\limits_{-\infty}^{\infty} f(m)\frac{m^{2k}}{(\Lambda_0)^{k+1}} e^{imx} dm$$ /31

$$(k = 0,\ 1,\ 2);$$

$$\varphi_3 = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(m)\,\varkappa(m,\ \alpha,\ \theta\varepsilon)\, e^{imx} dm;$$

$$\varkappa(m,\ \alpha,\ \varepsilon) = -\Lambda^{-4}(\Lambda')^3 + \Lambda^{-3}\Lambda'\Lambda'' - \frac{1}{6}\Lambda^{-1}\Lambda''', \quad 0 \leqslant \theta < 1;$$

$$\Lambda_0 = \alpha^2 + 4\omega^2 + m^2$$

(the prime designates differentiation with respect to ε).

Integrals (7) converge, but further computation is possible only for specific values of the function f(m). Let us now calculate the integrals ζ_k for an initial elevation of the form

$$\zeta(x,\ y,\ 0) = \begin{cases} a\cos\frac{\pi x}{2b} & |x| \leqslant b; \\ 0 & |x| > b. \end{cases}$$

In this case

$$f(m) = \frac{1}{b}\sqrt{\frac{\pi}{2}}\,\frac{\cos mb}{k^2 - m^2}, \tag{8}$$

where b and k are dimensionless quantities equal respectively to $b\sigma c^{-1}$,

$1/2\pi b^{-1}$.

Substituting expression (8) in equation (6) and subsequently computing the integrals $\phi_k(\alpha)$ and then ζ_k by the method of contour integration, we find expressions for the elevated liquid

$$\begin{gathered} 1)\quad x > ct + b \\ \xi(x, t) = 0; \\ 2)\quad ct - b < x < ct + b \\ \xi(x, t) = 1/2a(\eta_0 - \varepsilon_1\eta_1 - \varepsilon_1^2\eta_2) + R, \end{gathered} \tag{9}$$

where

$$\eta_0 = q^{-2}\left\{\cos\gamma + 4d^{1/2}\exp\left[-\frac{2\omega(x-b)}{c}\right]\right\} - 2bcK_0(x-b); \tag{10}$$

$$\eta_1 = A_0\sin\left(\gamma - \frac{\pi}{4}\right) + A_1 t\cos\left(\gamma - \frac{\pi}{4}\right) + 8b^{3/2}c^{1/2}K_1(x-b);$$

$$\eta_2 = 2\omega E(x-b) + \frac{1}{16}B_1\frac{b}{c}q^{-5}\left\{qB_1\left[\left(\frac{x-b}{c}\right)^2 - \frac{5}{4}q^{-1}t\,\frac{x-b}{c} - \right.\right.$$

$$\left.\left. - \frac{5}{8}q^{-2}t^2 + \frac{2bd}{\pi c}q^{-2}\frac{x-b}{c}\right]\sin\gamma + \left(A_2\frac{x-b}{c} + A_3 t\right)\cos\gamma\right\};$$ /32

$$\varepsilon_1 = \nu^{1/2}c^{1/2}b^{-1/2}h^{-1};$$

$$K_0(x) = 2Jm\left\{e^{2\omega it}\int_0^\infty \frac{\sqrt{\xi^2 - 4\omega i\xi}\,\exp(-\xi t)\,\mathrm{ch}\,(xc^{-1}\sqrt{\xi^2 - 4\omega i\xi})}{(\xi - 2\omega i)(\pi^2c^2 + 4b^2\xi^2 - 16b^2\omega i\xi)}\,d\xi\right\};$$

$$d = (4b\omega\pi^{-1}c^{-1})^2;\quad q = (1+d)^{1/2};\quad \gamma = 1/2\pi b^{-1}(x - c_1 t);\quad c_1 = cq.$$

$$K_1(x) = 4b^2\,\mathrm{Re}\left\{e^{2\omega it}\int_0^\infty \frac{(\xi^2 - 4\omega i\xi)[D_1(\xi) + D_2(\xi)]}{(\xi - 2\omega i)(\pi^2c^2 + 4b^2\xi^2 - 16b^2\omega i\xi)}\,e^{-\xi t}\,d\xi\right\};$$

$$A_0 = \frac{1}{4}\left(\frac{b}{2c}\right)^{1/2}q^{-4}\left[B_3 - B_0 + \frac{\sqrt{2}b}{\pi c}(q^3 - 3\sqrt{2}d)B_1 q^{-1}\right];$$

$$A_1 = \frac{1}{2\sqrt{2}}\left(\frac{b}{2c}\right)^{1/2}q^{-4}B_1;\quad B_0 = (p + 2\omega)^{-1/2} + (p - 2\omega)^{-1/2};$$

$$p = \frac{\pi c}{2b}q;$$

$$A_2 = 4\left(B_0 - \frac{1}{2}B_3\right) + B_1\frac{b}{\pi c}q^{-1}(-16 + 5q^3 + q^2 + 12q);$$

$$A_3 = 4q^{-1}\left(B_0 - \frac{1}{2}B_3\right) - B_1\frac{b}{\pi c}q^{-2}(18 - 5q^2 - 12q);$$

$$E(x) = \frac{b^2}{16\omega^2 b^2 + \pi^2 c^2} \cdot \frac{x}{2c}\left[\frac{\omega x}{c}\left(2\omega\frac{x}{c} - 1\right) + \omega t\left(2\omega\frac{x}{c} - 5\right) \times \right.$$
$$\left. \times (3 - 2\omega t) - \frac{85}{8}\right]\exp\left(-\frac{2\omega x}{c}\right);$$
$$B_n = (p - 2\omega)(p + 2\omega)^{-n/2} + (p + 2\omega)(p - 2\omega)^{-n/2} \quad (n = 1,3);$$
$$D_n = \left[\frac{1}{4\sqrt{\xi^2 - 2\omega i\xi}} - \frac{2b^2\sqrt{\xi^2 - 2i\omega\xi}}{\pi^2 c^2 + 4b^2\xi^2 - 16b^2\omega\xi i} + (-1)^n\frac{x}{c}\right] \times$$
$$\times \left[\frac{\xi}{\sqrt{\xi - 4\omega i}} + (-1)^n \frac{\xi - 4\omega i}{\sqrt{\xi}}\right]\exp\left[(-1)^n \cdot \frac{x}{c}\sqrt{\xi^2 - 2\omega i\xi}\right];$$
$$(n = 1,2)$$

3) $b < x < ct - b.$

The expression for ζ is found from formula (10); in this case

$$\eta_0 = 8d^{1/2}q^{-1}\,\mathrm{ch}\left(\frac{2\omega b}{c}\right)\exp\left(-\frac{2\omega x}{c}\right) - 2bc\,[K_0(x - b) + K_0(x + b)$$

$$\eta_1 = -8b^{3/4}c^{1/2}\,[K_1(x - b) + K_1(x + b)];$$
$$\eta_2 = 2\omega\,[E(x - b) + E(x + b)]$$ /33

Here, x, b, t and ω are the original dimensional quantities. Analysis shows that the residual term R of the series is of order ε^3 for small values of ε. Therefore, the resultant expression (10) gives the form of the free surface in the region $|x| > b$ for finite values of time t with an error of the order of ε^3 when $\varepsilon \to 0$.

Let us note that when viscosity is disregarded, the expression for ζ in the region $ct - b < x < ct + b$ may be written as implied by equation (10), in the form

$$\zeta(x, t) = \frac{1}{2}a\cos(x - c_1 t) + 0(\omega) \quad (11)$$

Where $0(\omega)$ includes waves with amplitudes which approach zero as

$\omega \to 0$. Comparing equations (11) with expressions (13) and (14) in [8] where the effect of viscosity on the fundamental wave was studied, we see that rotation tends to increase the velocity of motion of the fundamental wave $c_1 > c$ and has no appreciable effect on its amplitude with an increase in time t, whereas the viscosity acts in the opposite manner tending to reduce the velocity of the fundamental wave and having a considerable effect on its amplitude (the amplitude of the fundamental wave decreases with time according to an exponential law).

The problem studied above was for initial elevation depending on a single variable. Let us consider a similar problem for initial elevation depending on two variables.

Let us assume that

$$\zeta(x, y, 0) = af(x, y) \tag{12}$$

and that there are no perturbations in the liquid when t = 0. The problem reduces to solution of the system [7]

$$\frac{\partial u}{\partial t} - 2\omega v = -g\frac{\partial \zeta}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}; \quad \frac{\partial v}{\partial t} + 2\omega u = -g\frac{\partial \zeta}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2};$$
$$\frac{\partial \zeta}{\partial t} = -\int_{-h}^{0}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dz \tag{13}$$

with initial and boundary conditions (2), where the function f depends on two variables--x and y.

Using for the solution of the system (13) the Fourier transform with respect to the variables x and y, and the Laplace transform with respect to time t and satisfying the initial and boundary conditions, we get an expression for the form of the free surface:

$$\zeta(R, \gamma, t) = \frac{a}{2\pi}\int_0^\infty\int_0^{2\pi}\psi(r, \theta)\varphi(r, t)\exp[irR\cos(\theta - \gamma)]\, r\,dr\,d\theta; \tag{14}$$

$$\varphi(r) = \frac{1}{2\pi i}\int\limits_{s-i\infty}^{s+i\infty} \frac{\alpha^2 + 4\omega^2}{\alpha\Delta(r, \alpha, \varepsilon)} e^{\alpha t} d\alpha;$$

$$\psi(r, \theta) = \frac{1}{2\pi}\int\limits_0^\infty\int\limits_0^{2\pi} f(R, \gamma)\exp[-irR\cos(\theta - \gamma)]\, RdRd\gamma \qquad (15)$$

Here $\Delta(r, \alpha, \varepsilon)$ ε, σ, c are the same quantities as in equation (5); R, t and ω are dimensionless quantities equal respectively to $R\sigma c^{-1}$, σt, $\omega\sigma^{-1}$; polar coordinates R and γ are introduced in place of the values x and y.

Since integral (15) is convergent, $r\phi(r) \to 0$ when $r \to \infty$ and $\psi(r, \theta) \to$ $\to 0$ when $r \to \infty$, at least as r^{-1}, then integral (14) will obviously converge for an arbitrary function f(x, y) which may be represented by a Fourier integral. Expression (14) is an exact solution for the problem as formulated, valid for any values of the parameter ε.

Let us analyze expression (14) further for small values of the parameter ε. Expanding the function Δ^{-1} (r, α, ε) in a series with respect to the parameter ε, limiting ourselves to the first three terms of the series as in the preceding case, and writing the residual term of the series in Lagrangian form, we get expression (14) in the form

$$\zeta = \zeta_0 + \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \varepsilon^3\zeta_3; \qquad (16)$$

$$\zeta_k = \frac{a}{2\pi}\int\limits_0^\infty\int\limits_0^{2\pi} \psi(r, \theta)\,\varphi_k(r)\, r^{2k+1}\exp[irR\cos(\theta - \gamma)]\, drd\theta; \qquad (17)$$

$$\varphi_k = \frac{1}{2\pi i}\int\limits_{s-i\infty}^{s+i\infty} \frac{\alpha^2 + 4\omega^2}{[\alpha(\alpha^2 + 4\omega^2 + r^2)]^{k+1}}\left(\frac{\alpha - 2\omega i}{2\sqrt{\alpha + 2\omega i}} + \right.$$

$$\left. + \frac{\alpha + 2\omega i}{2\sqrt{\alpha - 2\omega i}}\right)^k e^{\alpha t} d\alpha \quad (k = 0, 1, 2); \qquad (18)$$

$$\varphi_3 = \frac{1}{2\pi i}\int\limits_{s-i\infty}^{s+i\infty} (\alpha^2 + 4\omega^2)\,\varkappa(r, \alpha, \theta\varepsilon)\, e^{\alpha t} d\alpha \quad (0 < \theta < 1),$$

where κ has the same value as in equation (7).

An investigation of the expression for ζ_3 shows that it is bounded and has a finite limit for finite values of time t as $\varepsilon \to 0$. Therefore, the first three terms of expression (16) give the form of the free surface of the liquid with an error of the order of ε^3 as $\varepsilon \to 0$.

For an initial elevation which is symmetric with respect to the coordinate origin [f(x, y) = f(R)], the form of the free surface will be expressed by formula (16), where ζ_k, in view of the equation /35

$$\frac{1}{2\pi}\int_0^{2\pi} \exp(irR\cos\theta)\,d\theta = J_0(rR)$$

takes the simpler form

$$\zeta_k = a\int_0^{\infty} \psi(r)\,\varphi_k(r)\,J_0(rR)\,r^{2k+1}dr \quad (k = 0,\ 1,\ 2)$$
$$\psi(r) = \int_0^{\infty} f(R)\,J_0(rR)\,RdR. \qquad (19)$$

Using the method of contour integration for calculating integrals ϕ_k, we get an expression for the form of the waves on the free surface

$$\zeta = \zeta_0 + \varepsilon\zeta_1 + \varepsilon^2\zeta_2,$$

where the ζ_k are computed from formulas (17) in the general case, and from formulas (19) axisymmetric case, and for the expression ψ_k (k = 0, 1, 2) take the following form:

$$\varphi_0(r) = \cos qt + 4\omega^2 q^{-1}(1 - \cos qt);$$
$$\varphi_1(r) = \psi_1\cos qt + \psi_2\sin qt + \psi_3 + \psi_4;$$
$$\varphi_2(r) = \psi_5\cos qt + \psi_6\sin qt + \psi_7 + \psi_8;$$
$$\psi_{1,2} = -\frac{1}{4\sqrt{2}}\,r^2 q^{-4}\left\{B_1[\mp r^2 t + (r^2 - 8\omega^2)\,q^{-1}] - \frac{1}{2}r^2(B_3 + 8\omega A)\right\}; \qquad (20)$$
$$\psi_3 = \omega^{3/2} r^2 q^{-1}(3 - 4\omega t);$$

$$\psi_4 = -\frac{r^2}{\pi}\,\mathrm{Re}\left\{e^{2\omega it}\int\limits_0^\infty \frac{\sqrt{\xi}\,(\xi - 4\omega i)^2}{(2\omega i - \xi)^2(r^2 + \xi^2 - 4\omega i\xi)^2}\,e^{-\xi t}\,d\xi\right\};$$

$$\psi_5 = r^4 E t; \quad \psi_6 = -r^4(D + Gt^2);$$

$$\psi_7 = 2\omega r^2 q^{-6}\left(\frac{13}{8} - 6q^{-2} - 3\omega t + \omega^2 t^2\right);$$

$$\psi_8 = -\frac{2r^4}{\pi}\,\mathrm{Re}\left\{e^{2\omega it}\sqrt{4\omega i - 1}\,\times\right.$$

$$\left.\times\int\limits_0^\infty \frac{\xi^2(\xi - 4\omega i)}{(2\omega i - \xi)^3(r^2 + \xi^2 - 4\omega i\xi)^2}\,e^{-\xi t}\,d\xi\right\};$$

$$A = (q + 2\omega)^{-3/2} - (q - 2\omega)^{-3/2};$$

$$B_k = (q + 2\omega)(q - 2\omega)^{-k/2} + (q - 2\omega)(q + 2\omega)^{-k/2};$$ /36

$$C_k = (q + 2\omega)^{-k/2} + (q - 2\omega)^{-k/2};$$

$$D = \frac{1}{4}B_1\left[a_2B_1 + 4a_1\left(C_1 - \frac{1}{2}B_3\right)\right] - \frac{1}{2}a_0\left[\left(C_1 - \frac{1}{2}B_3\right)^2 + \right.$$

$$\left. + \left(\frac{3}{4}B_5 - C_3\right)B_1\right];$$

$$E = \frac{1}{2}B_1\left[2a_0\left(C_1 - \frac{1}{2}B_3\right) - a_1B_1\right];$$

$$G = \frac{1}{32}r^2q^{-6}B_1^2;$$

$$a_0 = \frac{1}{8}r^2q^{-6}; \quad a_1 = \frac{1}{4}q^{-5}\left(-1 + \frac{9}{4}r^2q^{-2}\right);$$

$$a_2 = (2 - 3r^2q^{-2})\,q^{-5}; \quad q = \sqrt{r^2 + 4\omega^2}$$

Let us note that t and ω are dimensionless in these formulas, and that all notation differs from that used in formula (10).

Let us find the asymptotic expression of the form of the free surface for a symmetric initial elevation and large values of R. Using the formula

$$\left|J_0(x) - \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right)\right| < \frac{B}{x^{3/2}},$$

which is valid for all positive values, we get

$$\zeta_k = a\sqrt{\frac{2}{\pi R}}\int_0^\infty \psi(r)\,\varphi_k(r)\,r^{2k+1/2}\cos\left(rR - \frac{\pi}{4}\right)dr + 0\,(R^{-3/2}), \tag{21}$$

where B is some positive number; ϕ_k is defined by formula (20). Analyzing integrals (21) by the method of stationary phases for large values of R, we get the following asymptotic expression for the form of the free surface

$$\zeta = \begin{cases} aR^{-1}F(r) & R < ct; \\ 0\,(R^{-3/2}) & R > ct; \end{cases} \tag{22}$$

where

$$F(r) = \mu^{1/4}(\mu - 1)^{3/4}\,[\varkappa_1(r)\sin\alpha_1 + \varkappa_2(r)\cos\alpha_1];$$

$$\varkappa_1(r) = [1 - 4\omega^2 q^{-2} + \varepsilon\psi_1(r) + \varepsilon^2\psi_5(r)]\,r^{1/2};$$

$$\varkappa_2(r) = -[\varepsilon\psi_2(r) + \varepsilon^2\psi_6(r)]\,r^{1/2};$$

/37

$$\mu = tR^{-1};\quad \alpha_1 = r_1 R - \sqrt{r_1^2 - 4\omega^2 t};\quad r_1 = 2\omega(\mu^2 - 1)^{-1/2}$$

The resultant asymptotic formula is true for values of $\mu > 1$ which are small in comparison with R and for non-oscillating functions $\psi(r)$, for instance for the functions f(R) of the form exp (-kR) or exp$(-kR^2)$; R, t and ω are dimensionless in formula (21). It is readily shown from formula (19) $t \to \infty$ at any fixed point of R, the elevation of ζ will tend toward zero as t^{-2}. It is evident from equation (21) that the leading front of the fundamental perturbations for a symmetric initial elevation will move over the surface of a viscuous rotating liquid in the radial direction with velocity c.

For an ideal liquid with regard to rotation, the expression for ζ takes the form

$$\zeta = a\int_0^\infty \psi(r)\, rJ_0(rR)\cos\sqrt{r^2+4\omega^2}\, t\,dr.$$

Assuming that the parameter ω is small (the maximum value of ω for the earth is $7.3\cdot10^{-5}$), expanding $\cos\sqrt{r^2+4\omega^2}\,t$ in a series with respect to the parameter $4\omega^2$, limiting ourselves in this case to the first two terms of the series, and writing up the residual term in Lagrangian form, we get

$$\zeta = a\,[\eta_0 + 4\omega^2\eta_1] + O(\omega^4);$$

$$\eta_0 = \int_0^\infty \psi(r)\, rJ_0(rR)\cos rt\,dr;$$

$$\eta_1 = \int_0^\infty \psi(r)\left[r^{-1}(1-\cos rt) - \frac{1}{2}t\sin rt\right] J_0(rR)\,dr.$$

For the function f(R) of the form

$$f(R) = \begin{cases} (1-R^2b^{-2})^\mu & R \leqslant b \\ 0 & R > b \end{cases} \qquad \mu > 0$$

we have [1]

$$\psi(r) = 2^\mu b^{-\mu+1}\Gamma(\mu+1)\, r^{-\mu-1} J_{\mu+1}(br). \tag{23}$$

Taking account of the fact that [1]

$$\int_0^\infty r^{-\mu} J_{\mu+1}(br) J_0(rR) \begin{Bmatrix} \cos rt \\ r^{-1}\sin rt \end{Bmatrix} dr = 0$$

For R > t + b in this region (in dimensional quantities R > ct + b), we find an expression for the form of the free surface /38

$$\zeta = 4\omega^2 a \int_0^\infty r^{-1}\psi(r)(1-\cos rt) J_0(rR)\,dr,$$

where $\psi(r)$ is defined by formula (23).

The resultant expression is the perturbation of the free surface which precedes the arrival of the fundamental wave $\eta_0(R,t)$ and is caused exclusively by the effect of the Coriolis force.

Thus in the case of axisymmetric initial elevation of the free surface and in the case of a liquid of constant depth, both the effect of the Coriolis force (without regard to viscosity) and the effect of forces of viscosity [8] (without regard to the effect of Coriolis force) give rise to disturbances which precede the waves, i.e. perturbations of the free surface observed before arrival of the fundamental wave.

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MOTION OF A WING WITH DEFLECTED AILERONS CLOSE TO A BAFFLE

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ABSTRACT: The motion of a horizontal wing of large aspect ratio with deflected ailerons close to a solid baffle is studied. Simple analytical expressions are derived which may be used for evaluating the effect which proximity of the baffle has on the lift coefficient, the induced drag coefficient, the moment of banking and the moment of yawing.

The motion of a wing with deflected ailerons in an unbounded fluid has been studied by many authors [1, 2 and others], however, the effect of flow boundaries on the characteristics of such a wing has not been sufficiently investigated. /39

The effect of the free surface on the characteristics of a hydrofoil with deflected ailerons was studied by T. Nishiyama [4]; he showed that the proximity of the free surface reduces the lift and the banking moment, but increases the inductive drag and the yawing moment.

Simple analytical expressions are derived in this paper which may be used for evaluating the effect which proximity of a solid baffle has on the hydrodynamic characteristics of a wing with deflected ailerons.

The distribution of circulation along the span of a horizontal wing of large aspect ratio moving steadily with a velocity v_0 at a distance H_0 from a flat solid baffle (Fig. 1) is determined by the equation [3]

$$\bar{\Gamma}(\bar{y}) = \frac{a_0}{2\lambda(\bar{y})}\psi(\bar{y})\left\{\alpha(\bar{y}) + \frac{1}{2\pi}\int_{-1}^{+1}\bar{\Gamma}'(\bar{\eta})\left[\frac{1}{(\bar{y}-\bar{\eta})^2} - \frac{(\bar{y}-\bar{\eta})^2 - 4\bar{H}_0^2}{[(\bar{y}-\bar{\eta})^2 + 4\bar{H}_0^2]^2}\right]d\bar{\eta}\right\}. \quad (1)$$

Here a_0 is the tangent of the slope to the curve $C_y = C_y(\alpha)$ for an unbounded fluid; $\psi(\bar{y})$ is a function which accounts for the change in a_0 in a bounded flow; $\alpha(y)$ is the angle of attack of the wing; /40

$$\bar{\Gamma}(\bar{y}) = \frac{\Gamma(y)}{\lambda B v_0}; \quad \bar{y} = \frac{2y}{L}; \quad \bar{z} = \frac{2z}{B}; \quad \bar{H}_0 = \frac{2H_0}{L};$$

$\lambda(\bar{y})$ is the aspect ratio of the wing equal to L/B (L is the span and B is the chord of the wing).

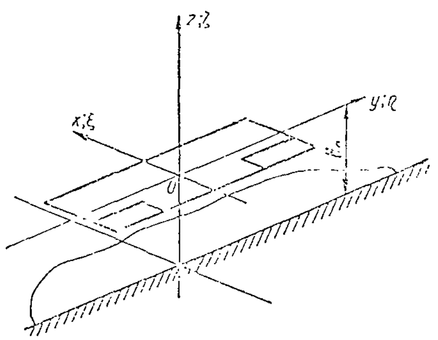

Fig. 1

Let us examine the case of motion of such a wing with ailerons deflected through a small angle. Let us assume that deflection of the ailerons through a small angle is equivalent to a certain twist of the wing. Deformation of the wing with deflection of the ailerons leads to asymmetric distribution of the circulation along the span. Let us represent the circulation as the sum of symmetric and asymmetric terms:

$$\bar{\Gamma}(\bar{y}) = \bar{\Gamma}_s(\bar{y}) + \bar{\Gamma}_a(\bar{y}) = \frac{\bar{\Gamma}(\bar{y}) + \bar{\Gamma}(-\bar{y})}{2} + \frac{\bar{\Gamma}(\bar{y}) - \bar{\Gamma}(-\bar{y})}{2}. \tag{2}$$

In an analogous manner, we write the angle of attack of the deformed wing

$$\alpha(\bar{y}) = \alpha_s(\bar{y}) + \alpha_a(\bar{y}) = \frac{\alpha(\bar{y}) + \alpha(-\bar{y})}{2} + \frac{\alpha(\bar{y}) - \alpha(-\bar{y})}{2}, \tag{3}$$

where $\alpha_s(\bar{y})$ is the angle of attack of the wing with undeflected ailerons; $\alpha_a(\bar{y})$ is the angle of twist of the wing which accounts for deflection of the ailerons.

The kernel of equation (1) may also be represented in the form of a symmetric and asymmetric parts:

$$G(\bar{y}, \bar{\eta}) = G_s(\bar{y}, \bar{\eta}) + G_a(\bar{y}, \bar{\eta}).$$

In a like manner

$$G_c(\bar{y}, \bar{\eta}) = G_{ss}(\bar{y}, \bar{\eta}) + G_{sa}(\bar{y}, \bar{\eta});$$

$$G_a(\bar{y}, \bar{\eta}) = G_{as}(\bar{y}, \bar{\eta}) + G_{aa}(\bar{y}, \bar{\eta}). \qquad (4)$$

/41

Analysis shows that in the given case

$$G_{sa}(\bar{y}, \bar{\eta}) = G_{as}(\bar{y}, \bar{\eta}) = 0. \qquad (5)$$

With regard to formulas (2)-(5), equation (1) may be reduced to two independent equations which determine the symmetric and the asymmetric parts of the circulation:

$$\Gamma_s(\bar{y}) = \frac{a_0}{2\lambda(\bar{y})}\psi(\bar{y})\left\{\alpha_s(\bar{y}) + \frac{1}{2\pi}\int_{-1}^{+1}\Gamma_c(\bar{\eta})\, G_{ss}(\bar{y}, \bar{\eta})\, d\bar{\eta}\right\}; \qquad (6)$$

$$\Gamma_a(\bar{y}) = \frac{a_0}{2\lambda(\bar{y})}\psi(\bar{y})\left\{\alpha_a(\bar{y}) + \frac{1}{2\pi}\int_{-1}^{+1}\Gamma_a(\bar{\eta})\, G_{aa}(\bar{y}, \bar{\eta})\, d\bar{\eta}\right\}, \qquad (7)$$

where
$$G_{ss}(\bar{y}, \bar{\eta}) = \frac{1}{2}\left\{\frac{1}{(\bar{y}-\bar{\eta})^2} - \frac{(\bar{y}-\bar{\eta})^2 - 4\bar{H}_0^2}{[(\bar{y}-\bar{\eta})^2 + 4\bar{H}_0^2]^2} + \frac{1}{(\bar{y}+\bar{\eta})^2} - \frac{(\bar{y}+\bar{\eta})^2 - 4\bar{H}_0^2}{[(\bar{y}+\bar{\eta})^2 + 4\bar{H}_0^2]^2}\right\};$$

$$G_{aa}(\bar{y}, \bar{\eta}) = \frac{1}{2}\left\{\frac{1}{(\bar{y}-\bar{\eta})^2} - \frac{(\bar{y}-\bar{\eta})^2 - 4\bar{H}_0^2}{[(\bar{y}-\bar{\eta})^2 + 4\bar{H}_0^2]^2} - \frac{1}{(\bar{y}+\bar{\eta})^2} + \frac{(\bar{y}+\bar{\eta})^2 - 4\bar{H}_0^2}{[(\bar{y}+\bar{\eta})^2 + 4\bar{H}_0^2]^2}\right\}.$$

When $\bar{H}_0 \to \infty$, equations (6) and (7) take the form

$$\bar{\Gamma}_S(\bar{y}) = \frac{a_0}{2\lambda(\bar{y})}\left\{\alpha_S(\bar{y}) + \frac{1}{4\pi}\int_{-1}^{+1}\bar{\Gamma}_S(\bar{\eta})\left[\frac{1}{(\bar{y}-\bar{\eta})^2} + \frac{1}{(\bar{y}-\bar{\eta})^2}\right]d\bar{\eta}\right\}; \tag{8}$$

$$\bar{\Gamma}_a(\bar{y}) = \frac{a_0}{2\lambda(\bar{y})}\left\{\alpha_a(\bar{y}) + \frac{1}{4\pi}\int_{-1}^{+1}\bar{\Gamma}_a(\bar{\eta})\left[\frac{1}{(\bar{y}-\bar{\eta})^2} - \frac{1}{(\bar{y}+\bar{\eta})^2}\right]d\bar{\eta}\right\}. \tag{9}$$

Let us give approximate solutions of equations (6)-(9) for a wing of elliptical planform. In conformity with this, we assume

$$\frac{\psi(\bar{y})}{\lambda(\bar{y})} = \frac{4\psi}{\pi\lambda_0}\sqrt{1-\bar{y}^2}. \tag{10}$$

We shall seek the solutions of the equations in the form

$$\bar{\Gamma}_S(\bar{y}) = A\sqrt{1-\bar{y}^2}.$$

$$\Gamma_a(\bar{y}) = B\bar{y}\sqrt{1-\bar{y}^2}, \tag{11}$$ /42

where A and B are constants.

Integrating by parts, we reduce equations (8) and (9) to integral differential form:

$$\bar{\Gamma}_S(\bar{y}) = \frac{a_0}{2\lambda(\bar{y})}\left\{\alpha_S(\bar{y}) - \frac{1}{4\pi}\int_{-1}^{+1}\bar{\Gamma}'_S(\eta)\left[\frac{1}{(\bar{y}-\eta)} - \frac{1}{(\bar{y}+\eta)}\right]d\bar{\eta}\right\}; \tag{12}$$

$$\bar{\Gamma}_a(\bar{y}) = \frac{a_0}{2\lambda(\bar{y})}\left\{\alpha_a(\bar{y}) - \frac{1}{4\pi}\int_{-1}^{+1}\bar{\Gamma}'_a(\bar{\eta})\left[\frac{1}{(\bar{y}-\eta)} + \frac{1}{(\bar{y}+\eta)}\right]d\bar{\eta}\right\}. \tag{13}$$

In solving equation (12), we take account of the fact that $\omega_c(\bar{y}) = \alpha$, where α is the angle of attack for the wing with undeflected ailerons.

The solution of this equation with regard to equations (10) and (11) gives

$$A_\infty = \frac{\frac{2a_0}{\pi\lambda_0}\,\alpha}{1+\frac{a_0}{\pi\lambda_0}}. \tag{14}$$

The solution of equation (13) with regard to expression (10) and (11) gives a linear law for distribution of the angle of twist for a deformed wing along the span

$$\alpha_a(\bar{y}) = \alpha_f \bar{y}, \tag{15}$$

Where α_f is the angle of twist for a finite cross-section of the deformed wing.

The constant B_∞ in this case is defined by the formula

$$B_\infty = \frac{\frac{2a_0}{\pi\lambda_0}\,\alpha_f}{1+\frac{2a_0}{\pi\lambda_0}}. \tag{16}$$

After integration by parts, equations (6) and (7) reduce to integral differential form:

$$\bar{\Gamma}_S(\bar{y}) = \frac{a_0}{2\lambda(\bar{y})}\,\psi(\bar{y})\left\{\alpha_S(\bar{y}) - \frac{1}{4\pi}\int_{-1}^{+1}\bar{\Gamma}'_S(\bar{\eta})\left[\frac{1}{\bar{y}-\bar{\eta}} - \frac{\bar{y}-\bar{\eta}}{(\bar{y}-\bar{\eta})^2+4\bar{H}_0^2} - \frac{1}{\bar{y}+\bar{\eta}} + \frac{\bar{y}+\bar{\eta}}{(\bar{y}+\bar{\eta})^2+4\bar{H}_0^2}\right]d\bar{\eta}\right\}; \tag{17}$$

$$\bar{\Gamma}_a(\bar{y}) = \frac{a_0}{2\lambda(\bar{y})}\,\psi(\bar{y})\left\{\alpha_a(\bar{y}) - \frac{1}{4\pi}\int_{-1}^{+1}\bar{\Gamma}'_a(\bar{\eta})\left[\frac{1}{\bar{y}-\bar{\eta}} - \frac{\bar{y}-\bar{\eta}}{(\bar{y}-\bar{\eta})^2+4\bar{H}_0^2} + \frac{1}{\bar{y}+\bar{\eta}} - \frac{\bar{y}+\bar{\eta}}{(\bar{y}+\bar{\eta})^2+4\bar{H}_0^2}\right]d\bar{\eta}\right\}. \tag{18}$$ /43

Solving these equations with regard to expressions (10) and (11) and assuming a linear law for distribution of the angle of twist of the wing along the span, we get the following formulas for determining the constants A_H and B_H:

$$A_H = \frac{\frac{2a_0\psi}{\pi\lambda_0}\alpha}{1+\frac{a_0\psi}{\pi\lambda_0}\zeta_1}; \tag{19}$$

$$B_H = \frac{\frac{2a_0\psi}{\pi\lambda_0}\alpha_f}{1+\frac{2a_0\psi}{\pi\lambda_0}\zeta_2}. \tag{20}$$

In these formulas

$$\zeta_1 = 1+\frac{1}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}d\bar{y}\int_{-1}^{+1}\frac{\bar{\eta}}{\sqrt{1-\bar{\eta}^2}}\left[\frac{\bar{y}-\bar{\eta}}{(\bar{y}-\bar{\eta})^2+4\bar{H}_0^2}-\right.$$
$$\left.-\frac{\bar{y}+\bar{\eta}}{(\bar{y}+\bar{\eta})^2+4\bar{H}_0^2}\right]d\bar{\eta}; \tag{21}$$

$$\zeta_2 = 1-\frac{2}{\pi^2}\int_{-1}^{+1}\bar{y}\sqrt{1-\bar{y}^2}\,d\bar{y}\int_{-1}^{+1}\left[\frac{1}{\sqrt{1-\eta^2}}-\frac{2\eta^2}{\sqrt{1-\eta^2}}\right]\times$$
$$\times\left[\frac{\bar{y}-\eta}{(\bar{y}-\eta)^2+4H_0^2}+\frac{\bar{y}+\eta}{(\bar{y}+\eta)^2+4H_0^2}\right]d\eta. \tag{22}$$

We find the lift coefficient, the coefficient of inductive drag, the banking moment and the yawing moment from the expressions

$$C_y = \lambda_0\int_{-1}^{+1}\bar{\Gamma}_S(\bar{y})\,d\bar{y};$$

$$C_{xi} = \lambda_0\int_{-1}^{+1}[\bar{v}_{iS}\bar{\Gamma}_S(\bar{y})+\bar{v}_{ia}\bar{\Gamma}_a(\bar{y})]d\bar{y};$$

/44

$$\left.\begin{aligned} C_{mx} &= \frac{1}{2}\lambda_0 \int_{-1}^{+1} \bar{y}\Gamma_a(\bar{y})\, d\bar{y}; \\ C_{mz} &= \frac{1}{2}\lambda_0 \int_{-1}^{+1} \bar{y}\,[\bar{v}_{is}\Gamma_a(\bar{y}) + \bar{v}_{ia}\Gamma_S(\bar{y})]\, d\bar{y}. \end{aligned}\right\} \tag{23}$$

Here $\bar{v}_{is}$ and $\bar{v}_{ia}$ are the symmetric and asymmetric parts of the inductive velocity $\bar{v}_i$. For these quantities we have the expressions

$$\left.\begin{aligned} \bar{v}_{is\infty} &= -\frac{1}{4\pi}\int_{-1}^{+1}\Gamma_S(\bar{\eta})\left[\frac{1}{(\bar{y}-\bar{\eta})^2}+\frac{1}{(\bar{y}+\bar{\eta})^2}\right]d\bar{\eta}; \\ \bar{v}_{ia\infty} &= -\frac{1}{4\pi}\int_{-1}^{+1}\Gamma_a(\bar{\eta})\left[\frac{1}{(\bar{y}-\bar{\eta})^2}-\frac{1}{(\bar{y}+\bar{\eta})^2}\right]d\bar{\eta}; \\ \bar{v}_{is_H} &= -\frac{1}{4\pi}\int_{-1}^{+1}\Gamma_S(\bar{\eta})\left\{\frac{1}{(\bar{y}-\bar{\eta})^2}-\frac{(\bar{y}-\bar{\eta})^2-4\bar{H}_0^2}{[(\bar{y}-\bar{\eta})^2+4\bar{H}_0^2]^2}+\right. \\ &\left.+\frac{1}{(\bar{y}+\bar{\eta})^2}-\frac{(\bar{y}+\bar{\eta})^2-4\bar{H}_0^2}{[(\bar{y}+\bar{\eta})^2+4\bar{H}_0^2]^2}\right\}d\bar{\eta}; \\ \bar{v}_{is_H} &= -\frac{1}{4\pi}\int_{-1}^{+1}\Gamma_a(\bar{\eta})\left\{\frac{1}{(\bar{y}-\bar{\eta})^2}-\frac{(\bar{y}-\bar{\eta})^2-4\bar{H}_0^2}{[(\bar{y}-\bar{\eta})^2+4\bar{H}_0^2]^2}-\right. \\ &\left.-\frac{1}{(\bar{y}+\bar{\eta})^2}+\frac{(\bar{y}+\bar{\eta})^2-4\bar{H}_0^2}{[(\bar{y}+\bar{\eta})^2+4\bar{H}_0^2]^2}\right\}d\bar{\eta}. \end{aligned}\right\} \tag{24}$$

Substituting expressions (11) and (24) in expression (23), and also taking account of formulas (14)(16)(19) and (20), we get the following formulas for the characteristics of a wing with deflected ailerons:

for motion in an unbounded fluid

$$\left.\begin{aligned} C_{y\infty} &= \frac{a_0\alpha}{1+\frac{a_0}{\pi\lambda_0}}; \\ C_{mx\infty} &= \frac{\frac{1}{8}a_0\alpha_K}{1+\frac{2a_0}{\pi\lambda_0}}; \end{aligned}\right\} \tag{25}$$

$$C_{xi\infty} = \frac{1}{\pi\lambda_0}\left[C^2_{y\infty} + 32 C^2_{mx\infty}\right];$$
$$C_{mz\infty} = \frac{3}{\pi\lambda_0} C_{y\infty} C_{mx\infty};$$ /45

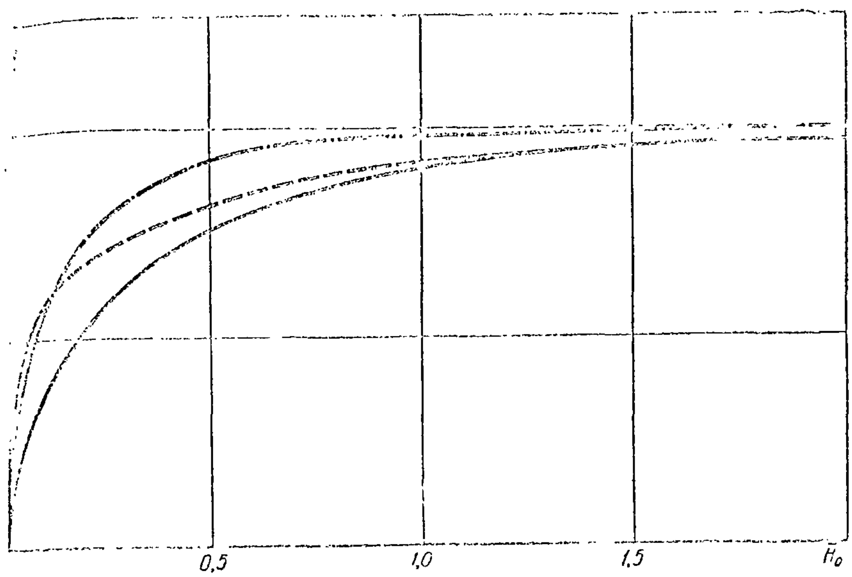

Fig. 2.

for motion close to a solid baffle

$$\left.\begin{aligned}
C_{yH} &= \frac{a_0\psi\alpha}{1 + \frac{a_0\psi}{\pi\lambda_0}\zeta_1};\\
C_{mxH} &= \frac{\frac{1}{8} a_0\psi\alpha_\kappa}{1 + \frac{2a_0\psi}{\pi\lambda_0}\zeta_2};\\
C_{xiH} &= \frac{1}{\pi\lambda_0}\left[C^2_{yH}\zeta_1 + 32\, C^2_{mxH}\zeta_2\right];\\
C_{mzH} &= \frac{1}{\pi\lambda_0} C_{yH} C_{mxH}\left[2\zeta_2 + \zeta_3\right].
\end{aligned}\right\} \quad (26)$$

The function ζ_3 is determined from the expression

$$\zeta_3 = 1 + \frac{4}{\pi^2}\int_{-1}^{+1} \bar{y}^2\sqrt{1-\bar{y}^2}\,d\bar{y}\int_{-1}^{+1}\frac{\bar{\eta}}{\sqrt{1-\bar{\eta}^2}}\times$$

$$\times\left[\frac{\bar{y}-\bar{\eta}}{(\bar{y}-\bar{\eta})^2+4\bar{H}_0^2}-\frac{\bar{y}+\bar{\eta}}{(\bar{y}+\bar{\eta})^2+4\bar{H}_0^2}\right]d\bar{\eta}. \qquad (27)$$

/46

The coefficients for the effect which proximity of a solid baffle has on the hydrodynamic characteristics of a wing with deflected ailerons are determined from the formulas

$$\bar{p}_y = \frac{C_{yH}}{C_{y\infty}} = \frac{\psi\left(1+\frac{a_0}{\pi\lambda_0}\right)}{\left(1+\frac{a_0\psi}{\pi\lambda_0}\zeta_1\right)};$$

$$\bar{m}_x = \frac{C_{mxH}}{C_{mx\infty}} = \frac{\psi\left(1+\frac{2a_0}{\pi\lambda_0}\right)}{\left(1+\frac{2a_0\psi}{\pi\lambda_0}\zeta_2\right)};$$

$$\bar{p}_{xi} = \frac{C_{xi_H}}{C_{xi\infty}} = \bar{p}_y^2\,\frac{\left[\zeta_1+\frac{1}{2}\frac{\left(1+\frac{a_0\psi}{\pi\lambda_0}\zeta_1\right)^2}{\left(1+\frac{2a_0\psi}{\pi\lambda_0}\zeta_2\right)^2}\cdot\frac{\alpha_f^2}{\alpha^2}\zeta_2\right]}{\left[1+\frac{1}{2}\frac{\left(1+\frac{a_0}{\pi\lambda_0}\right)^2}{\left(1+\frac{2a_0}{\pi\lambda_0}\right)^2}\cdot\frac{\alpha_f^2}{\alpha^2}\right]};$$

$$\bar{m}_z = \frac{C_{mzH}}{C_{mz\infty}} = \frac{1}{3}\bar{p}_y\bar{m}_x\,[2\zeta_2+\zeta_3].$$

The function ψ may be determined from the formula

$$\psi = 1 + \tau^2 + 0\,(\tau^1),$$

where

$$\tau = \sqrt{\lambda^2 \bar{H}_0^2 + 1} - \lambda \bar{H}_0.$$

The following approximate formulas are derived for calculating the functions ζ_1, ζ_2 and ζ_3:

$$\zeta_1 = 1 - 0{,}5\tau^2 - 0{,}25\tau^4 - 0{,}0625\tau^6 - 0{,}0468\tau^8 - 0{,}0233\tau^{10} - 0{,}0199\tau^{12};$$

$$\zeta_2 = 1 - 0{,}375\tau^4 - 0{,}250\tau^6 - 0{,}780\tau^8 - 0{,}0624\tau^{10} - 0{,}0340\tau^{12};$$

$$\zeta_3 = 1 - 0{,}5\tau^2 + 0{,}125\tau^4 + 0{,}0312\tau^6 - 0{,}1408\tau^8 - 0{,}0692\tau^{10} - 0{,}0700\tau^{12} - 0{,}0356\tau^{14}.$$

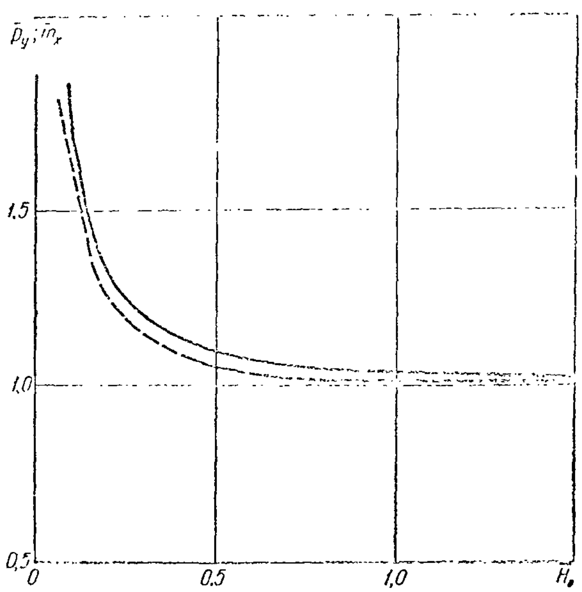

Fig. 3.

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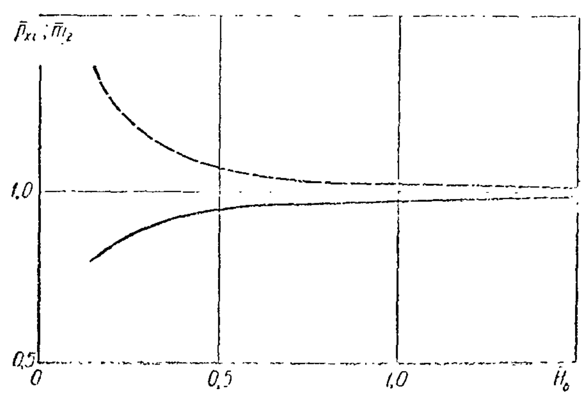

Fig. 4.

The graphs of functions ζ_1 (solid), ζ_2 (dot-and-dash) and ζ_3 (broken) are given in Fig. 2 and the /48 graphs of functions $\bar{p}_y$, $\bar{p}_{xi}$ (solid lines) and $\bar{m}_x$, $\bar{m}_z$ (broken lines) are given in Figure 3 and 4.

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LAMINAR BOUNDARY LAYER ON A HEAT-INSULATING SURFACE AT HIGH GAS VELOCITIES WITH SUCTION

L. F. Kozlov

ABSTRACT: A simple approximate method is proposed for calculating a laminar boundary layer in the case of an arbitrary velocity field on the external boundary with suction. The case of a heat-insulating surface is considered with a Prandtl number equal to Unity. The steady-state motions of a wing and a solid of revolution are analyzed in a compressible gas at Mach numbers of less than three.

The suction of a gas through the permeable surface of a wing or a solid of revolution moving at high velocities may be used for streamlining the flow or for preventing detachment of the boundary layer. Practical use of such a method for controlling the boundary layer may give an appreciable improvement in the aerodynamic characteristics of aircraft and vehicles moving close to the surface of the ground or water by reducing the components of frictional drag and pressure. /49

Solutions are known for the system of equations of a laminar boundary layer in a compressible gas for the special case of a plate with uniform suction [6, 7]. However, these solutions have more theoretical than practical significance since the necessity for integrating the boundary layer equation for the most part arises for the case of arbitrary velocity distribution on the external boundary.

Existing methods for approximate integration of laminar boundary layer equations are based on the use of the integral relationship of impulses. The accuracy of these methods depends to a considerable extent on successful approximation of the velocity profile across the layer. For instance, the velocity profile proposed by G. Schlichting and used in [8] for computing the laminar boundary layer in a compressible gas gives a very rough approximation of the actual change in velocities in the boundary layer, especially near the point of detachment of the layer.

Let us examine steady streamline flow of a compressible gas at high velocities around a flat wing. For this purpose, we introduce a coordinate system with the origin located at the leading critical point, the x-axis directed /50
along the surface and the y-axis directed along the normal to the wing surface. Assuming a Prandtl number equal to unity, we write the system of differential equations for the laminar boundary layer with regard to longitudinal pressure

gradient on the outer boundary in the form

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp_\delta}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right); \tag{1}$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0; \tag{2}$$

$$\rho u \frac{\partial \theta}{\partial x} + \rho v \frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial \theta}{\partial y}\right); \tag{3}$$

$$p_\delta = \rho R T; \tag{4}$$

$$\mu = \mu_1 \left(\frac{T}{T_1}\right)^n. \tag{5}$$

Here, u and v are the projections of the velocity vector in the boundary layer on the coordinate axes; p_δ is the predetermined pressure distribution on the outer boundary of the layer; θ is the stagnation temperature, equal to $T + (u^2/2Jc_p)$; T is absolute temperature; J is the mechanical equivalent of heat[1]; R is the gas constant equal to $J(c_p - c_v)$; c_p, c_v are the coefficients of specific heat of the gas at constant pressure and constant volume, respectively; μ is the dynamic coefficient of viscosity; ρ is the density of the gas. We shall use the subscript 1 to indicate the values of these same quantities in adiabatically and isentropically decelerated flows, and the subscript δ to indicate the values of the quantities on the outer surface of the boundary layer.

For air R = 287.1 m^2/deg·product sec^2, n = 0.75 at temperature $-23° < T < 327°C$.

System of equations (1)-(5) is closed and therefore may be used for determining the following unknown quantities: The two components of velocity, viscosity, density and temperature of the gas.

We satisfy the following boundary conditions on the heat-insulated wing surface in the case of suction, and on the outer boundary of the layer

[1]In the International System of Units J = 1.

$$u = 0; \quad v = v_0; \quad \frac{\partial T}{\partial y} = 0 \text{ when } y = 0;$$
$$u = U_\delta; \quad T = T_\delta \text{ when } y = \delta,$$

Where v_0 is the localized suction velocity; U_δ and T_δ are the velocity and temperature of the gas respectively on the outer boundary of the layer; δ is the thickness of the boundary layer.

Let us consider the case of a heat-insulating surface, i.e. where there is no heat transfer through the surface. In this case, the heat will be carried by convection due to suction of the gas through the porous surface. /51

The integral of energy balance equation (3) is obvious[1] [4]:

$$0 = \text{const} \quad \text{or} \quad T + \frac{u^2}{2Jc_p} = \frac{i_0}{Jc_p}, \tag{6}$$

where $i_0 = Jc_p T_\delta + \frac{U_\delta^2}{2}$ is the total energy.

Integral (6) satisfies the condition for absence of heat transfer through the surface since the kinematic condition of gas attachment to the surface of the foil is fulfilled. It follows from this integral that the surface temperature is equal to the stagnation temperature of the oncoming flow when there is no heat transfer.

We find from equation (6) that

$$T = T_1\left(1 - \frac{u^2}{2i_0}\right), \text{ where } T_1 = \frac{i_0}{Jc_p}\left(T_\delta + \frac{U_\delta^2}{2}\right). \tag{7}$$

According to expression (7), the temperature depends only on the change in the longitudinal component of velocity. Therefore, the form of the boundary conditions for the transverse component of velocity on the surface has no effect on this expression.

[1] This special solution is valid for streamline flow with a longitudinal pressure gradient [9].

We compute the pressure from Bernoulli's equation:

$$p = p_{\delta 1}\left(1 - \frac{U_\delta^2}{2i_0}\right)^{\frac{\varkappa}{\varkappa-1}}, \tag{8}$$

while the relationship for density is determined from the ideal gas law

$$\varrho = \varrho_{\delta 1}\left(1 - \frac{U_\delta^2}{2i_0}\right)^{\frac{\varkappa}{\varkappa-1}} \Big/ \left(1 - \frac{u^2}{2i_0}\right). \tag{9}$$

In equations (8)-(9), $\kappa = c_p/c_v$.

Using power relationship (5), we get

$$\mu = \mu_1\left(1 - \frac{u^2}{2i_0}\right)^n. \tag{10}$$

Our subsequent calculations are done in Dorodnitsyn's variables [1], which in the given case take the form

$$\xi = \int_0^x (1 - \bar{U}^2)^{\frac{\varkappa}{\varkappa-1}} dx; \tag{11}$$

$$\eta = \int^y \frac{(1 - \bar{U}^2)^{\frac{\varkappa}{\varkappa-1}}}{(1 - \bar{u}^2)} dy. \tag{12}$$ /52

Then the equation of the boundary layer (1) and the equation of continuity (2) are transformed to give

$$u\frac{\partial u}{\partial \xi} + w\frac{\partial u}{\partial \eta} = \frac{1 - \bar{u}^2}{1 - \bar{U}^2} U_\delta U_\delta' + \nu_{\delta 1}\left[(1 - \bar{u}^2)^{n-1}\frac{\partial u}{\partial \eta}\right]; \tag{13}$$

$$\frac{\partial u}{\partial \xi} + \frac{\partial w}{\partial \eta} = 0, \tag{14}$$

where

$$U'_\delta = \frac{dU_\delta}{d\xi}; \qquad w = \frac{v}{1-\overline{u}^2} + \frac{u}{(1-\overline{U}^2)^{\frac{\varkappa}{\varkappa-1}}} \cdot \frac{d\eta}{dx}.$$

After transformation of equation (13) with the application of expression (14) and integration by terms with respect to η from 0 to δ_η, we get the integral relationship

$$\frac{d}{d\xi}\int_0^{\delta_\eta} u(U_\delta - u)\,d\eta + \int_0^{\delta_\eta} \frac{\partial}{\partial\eta}[w(U_\delta - u)]\,d\eta -$$
$$- U' \int_0^{\delta_\eta} \left(\frac{1-\overline{u}^2}{1-\overline{U}^2}U_\delta - u\right) d\eta = \nu_{\delta 1}\left(\frac{\partial u}{\partial \eta}\right)_{\eta=0}, \tag{15}$$

where δ_η is the thickness of the boundary layer with respect to coordinate η.

Since we have $\overline{u} = 0$ when $\eta = 0$, the integral

$$\int_0^{\delta_\eta} \frac{\partial}{\partial\eta}[w(U_\delta - u)]\,d\eta = U_\delta v_0. \tag{16}$$

Taking account of expression (16), we transform integral relationship (15) to

$$\frac{df}{d\xi} - f\frac{d}{d\xi}\ln\frac{U'_\delta}{(1-\overline{U}^2)^2} = [F(f) - t^{**}]\frac{d}{d\xi}\ln\frac{U_\delta}{\sqrt{1-\overline{U}^2}}. \tag{17}$$

where

$$f = \frac{U_\delta' \delta^{**2}}{\nu_{\delta 1}(1-\bar{U}^2)};$$

(18)

$$t^{**} = \frac{v_0 \delta_\eta^{**}}{\nu_{\delta 1}};$$

(19)

$$F(f, t^{**}) = 2\{\zeta(f, t^{**}) - [2 + H(f, t^{**})] f\}; \quad (20)$$ /53

$$\zeta(f, t^{**}) = \frac{\delta_\eta^{**}}{U_\delta}\left(\frac{\partial u}{\partial \eta}\right)_{\eta=0}; \quad (21)$$

$$H(f, t^{**}) = \frac{\delta_\eta^*}{\delta_\eta^{**}}; \quad (22)$$

$$\delta_\eta^* = \int_0^\eta \left(1 - \frac{u}{U_\delta}\right) d\eta; \quad (23)$$

$$\delta_\eta^{**} = \int_0^{\delta_\eta} \frac{u}{U_\delta}\left(1 - \frac{u}{U_\delta}\right) d\eta. \quad (24)$$

Returning to the physical x-y plane, we reduce integral expression (17) to the final form

$$\frac{df}{dx} - f \frac{d}{dx} \ln \frac{dU_\delta/dx}{(1-\bar{U}^2)^{2+\frac{\varkappa}{\varkappa-1}}} = [F(f) - t^{**}] \frac{d}{dx} \ln \frac{U_\delta}{\sqrt{1-\bar{U}^2}}. \quad (25)$$

In order to calculate relationships (18)-(24), we assume that the velocity profiles across the boundary layer are a single-parameter family:

$$\frac{u}{U_\delta} = \varphi(\eta / \delta_\eta^{**}, f, t^{**}) \quad (26)$$

and that they are independent of the Mach number in explicit form. For this purpose we use a six-degree polynomial [3] or a system of integral relationships of "three moments [2]. In both approaches to computation of the function, we get

$$F(f, t^{**}) = A(t^{**}) - B(t^{**})f, \tag{27}$$

/54

where the numerical values of A and B are given as a function of the suction parameter t** in Figures 1 and 2.

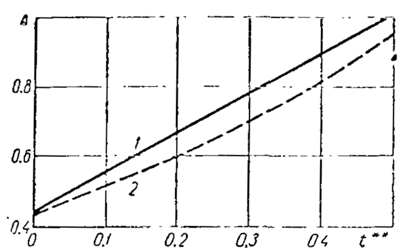

Fig. 1. Graph for A as a Function of the Suction Parameter t** : 1, From Data of [2]; 2, From Data of [3].

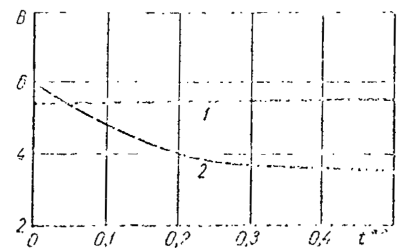

Fig. 2. Graph for B as a Function of the Suction Parameter t**: 1, From Data of [2]; 2, From Data of [3].

We integrate differential equation (25) in final form

$$f = \frac{dU_\delta/dx}{U_\delta^B (1 - \bar{U}^2)^N} \int_0^x U_\delta^{B-1} [A - 2t^{**}] (1 - \bar{U}^2)^{N-1} dx, \tag{28}$$

where

$$N = 2 + \frac{\varkappa}{\varkappa - 1} - \frac{B}{2}.$$

The values of the form parameter at the leading critical point f(0) = = A/B are shown in Figure 3, while the values of N are shown in Figure 4.

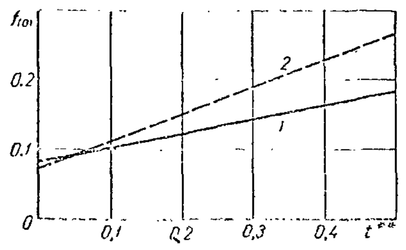

Fig. 3. Curves Which Show the Values of the Form Parameter at the Leading Critical Point f(0): 1, From Data of [2]; 2, From Data of [3].

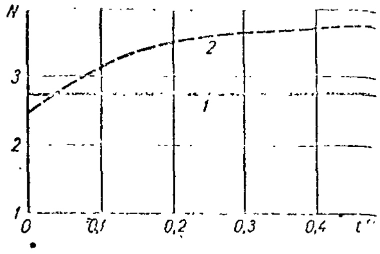

Figure 4. Graph for N in Air As a Function of the Suction Parameter t**: 1, From Data of [2]; 2, From Data of [3].

After calculating the form parameter f(x) from formula (28), we use the relationships $\zeta(f, t^{**})$ and $H(f, t^{**})$ given in [2 and 3] to determine all remaining characteristics of the laminar boundary layer in a compressible gas in the presence of suction from formulas (18)-(24).

The proposed method is developed in application to two-dimensional flow. In the case of an axisymmetric boundary layer on a solid of revolution, the same formulas may be used as are used for two-dimensional flow. The only change is in the form of equation of continuity (2) where the cofactor r_0, the radius of the cross-section of the solid of revolution, is added in parentheses. In this case, integral (28) takes on the form

$$f(x) = \frac{dU_\delta/dx}{U_\delta^B r_0^2 (1-\bar{U}^2)^N} \int_0^x U^{B-1} r_0^2 [A - 2t^{**}] (1-\bar{U}^2)^{N-1} dx. \quad (29)$$

The remaining characteristics of the laminar boundary layer on a solid of revolution may be calculated from the same formulas as those used for the two-dimensional case.

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EQUATIONS OF MOTION OF NON-NEWTONIAN FLUIDS

I. A. Pishchenko, O. M. Yakhno and A. I. Ovsyannikov

ABSTRACT: New equations of motion for non-Newtonian Fluids are theoretically analyzed with regard to their rheological characteristics (effect of transverse viscosity, etc.). The equations of motion for these fluids are given in both cylindrical and Cartesian coordinate systems.

It may be assumed in most cases that the motion of rheological (non-Newtonian) fluids in which consideration is given to the change in structure of the fluid itself caused by singularities in the given flow. These singularities may be due to processes of reorientation or some complication in the structure of extremely large molecules. For instance, during the motion of polymer media, straightening and stretching of the long-chain molecules is observed which leaves its imprint on the singularities of this flow. /56

The given structural changes in the flowing media are dependent to a considerable extent on the forces which produce this flow, i.e. on pressure forces. Depending on the time of application of these forces, the liquids may show various properties: elasticity, plasticity, etc., i.e. properties which are characteristics not only of non-Newtonian fluids, but also of Newtonian fluids and solids.

The development of the science of structure formation has shown that between the limiting states of matter--ideally elastic bodies and media which conform to Newton's law--there are continuous transitions which result in a tremendous variety of media of an intermediate nature which are essentially different from solids only in their relaxation properties.

Some authors [1, 3] have concluded that there are no fundamental differences between liquids and solids, and that there is a common constant which characterizes both types of media. This constant is the relaxation time

$$T = \frac{\eta}{E}$$

where η is the coefficient of viscosity and E is the modulus of elasticity for the medium.

If the deforming forces act on the given medium for a short period of time, any fluid will be like an ideal elastic medium. This statement is /57 completely applicable even to such classical Newtonian fluids as water. Actually, if the deforming forces act for a time period shorter than the relaxation time for water, which is $T_w = 10^{-13}$ sec, then the water will behave like an elastic body with a modulus of elasticity $E = 10^{11}$ N/m^2.

For considerably more viscous fluids such as polymer melts, the relaxation time may reach an appreciable value. In this case it becomes necessary to consider the elastic properties of the moving fluid. These properties depend to a considerable extent on the temperature of the medium since the values of T and η change considerably with a variation in temperature, while E is nearly constant.

The generality of the properties of rheological fluids is not limited to elastic properties alone. It is shown in [6] that certain conditions in highly viscous media (polymer media) may lead not only to stresses which are longitudinal with respect to the motion of the medium, but to stresses which are normal to that motion.

Weissenberg observed the "transverse viscosity effect" which now bears his name. This effect is observed under the condition

$$\frac{P_{an} - P_{bv}}{R_{an} - R_{bv}} = \frac{P_{bv} - P_{cw}}{R_{bv} - R_{cw}} = \frac{P_{cw} - P_{an}}{R_{cw} - R_{an}} = g(P, R), \tag{1}$$

where P_{an}, P_{bv}, P_{cw} are the principal Weissenberg stresses; R_{an}, R_{bv}, R_{cw} are the components of reversible deformation; g(P, R) is the shear modulus of the system.

Subsequent investigations have shown that this effect is observed not only in high-polymer solutions, but in any dispersed systems which are somewhat elastic with respect to shape. Since the elastic properties of a material are determined to a considerable extent by relaxation time, the extent to which the Weissenberg effect is observed also depends on time T.

The properties of rheological media given above are expressed in the equation of a state for a non-Newtonian fluid proposed by Reiner with regard to the hypothesis of existence of a coefficient of transverse viscosity η_w:

$$P_{ij} = -p\delta_{ij} + 2\eta R_{ij} + 4\eta_w \sum_{\alpha=1}^{3} R_{i\alpha} R_{\alpha j}, \tag{2}$$

where p is the internal pressure of the fluid; P_{ij} are the components of the stress tensor; δ_{ij} is the Kronecker delta; R_{ij}, $R_{i\alpha}$, $R_{\alpha j}$ are the respective components of the strain rate tensor; η, η_e are the coefficients of longitudinal and transverse viscosity, respectively.

According to formula (2), the tangential stress τ during motion of a rheological flow should be determined from the relationship /58

$$\tau = \eta\bar{R} + \frac{1}{2}\eta_e(\bar{R}\cdot\bar{R}), \tag{3}$$

where $\bar{R}$ is the strain rate tensor.

In this case it may be assumed that the coefficients of longitudinal and transverse viscosity are functions of the shear state. Since η and η_e are scalar quantities, they may be functions of scalar quantities only such as the three-invariants of the strain rate tensor. In view of the condition of continuity, the first invariant $I_1 = 0$. Consequently, η and η_e are functions of I_2 and I_3 only.

Let us assume in the first approximation that the quantity η_e is close to constant, and that the coefficient of longitudinal viscosity η may be determined from the Ostwald power law:

$$\eta = K\left[\frac{1}{2}I_2\right]^{\frac{n-1}{2}} \tag{4}$$

where K is a constant which characterizes the density of the medium; n is the flow index; I_2 is a quadratic invariant of the strain rate tensor $\bar{R}$.

In Cartesian coordinates, the expression for the second invariant of the strain rate tensor may be represented as

$$I_2 = 2\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right] + \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]^2 + \left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right]^2 + \left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right]^2. \tag{5}$$

For the case of two-dimensional flow, the given invariant may be rewritten as follows

$$I_2 = 2\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2\right] + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2. \tag{6}$$

In view of the fact that the change in velocity along the flow is considerably less pronounced than the change with respect to cross-section, it is natural to assume that relationships of the form

$$\frac{\dfrac{\partial v_x}{\partial x}}{\dfrac{\partial v_x}{\partial y}} \quad \text{and} \quad \frac{\dfrac{\partial v_x}{\partial x}}{\dfrac{\partial v_y}{\partial y}}$$

are vanishingly small and consequently

$$I_2 \approx \left(\frac{\partial v_x}{\partial y}\right)^2. \tag{7}$$

Substituting equation (4) with regard to expression (7) in relationship (2), we get an expression for the rheological Reiner's law /59

$$P_{ij} = p\delta_{ij} + 2K\,|I_2|^{\frac{n-1}{2}} R_{ij} + 4\eta_2 \sum_{\alpha=1}^{3} R_{i\alpha} R_{\alpha j}. \tag{8}$$

To derive an equation for the motion of a non-Newtonian fluid, we use the Reiner formulas derived above on the one hand, and equations of motion for a continuous medium on the other.

The equations of motion for a continuous medium are written in stresses in combination with the equation of continuity in the Cartesian coordinate system as follows:

$$\left.\begin{aligned}
&\varrho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=\\
&=-\frac{\partial p}{\partial x}+\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)+\varrho g_x;\\
&\varrho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=\\
&=-\frac{\partial p}{\partial y}+\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right)+\varrho g_y;\\
&\varrho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=\\
&=-\frac{\partial p}{\partial z}+\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right)+\varrho g_z;
\end{aligned}\right\}\qquad(9)$$

$$\frac{\partial\varrho}{\partial t}+\frac{\partial}{\partial x}(\varrho v_x)+\frac{\partial}{\partial y}(\varrho v_y)+\frac{\partial}{\partial z}(\varrho v_z)=0.\qquad(10)$$

According to formula (8), the relationship between the stress components and strain rate is written in the form

$$\left.\begin{aligned}
&\tau_{xx}=-p+2KI_2^{\frac{n-1}{2}}\cdot\frac{\partial v_x}{\partial x}+4\eta_e\sum_{\alpha=x,y,z}\frac{\partial v_\alpha}{\partial x}\cdot\frac{\partial v_x}{\partial\alpha};\\
&\tau_{yy}=-p+2KI_2^{\frac{n-1}{2}}\frac{\partial v_y}{\partial y}+4\eta_e\sum_{\alpha=x,y,z}\frac{\partial v_y}{\partial\alpha}\cdot\frac{\partial v_\alpha}{\partial y};\\
&\tau_{zz}=-p+2KI_2^{\frac{n-1}{2}}\frac{\partial v_z}{\partial z}+4\eta_e\sum_{\alpha=x,y,z}\frac{\partial v_z}{\partial\alpha}\cdot\frac{\partial v_\alpha}{\partial z};\\
&\tau_{xy}=KI_2^{\frac{n-1}{2}}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)+4\eta_e\sum_{\alpha=x,y,z}\left(\frac{\partial v_x}{\partial\alpha}+\frac{\partial v_\alpha}{\partial x}\right)\left(\frac{\partial v_\alpha}{\partial y}+\frac{\partial v_y}{\partial\alpha}\right);\\
&\tau_{xz}=KI_2^{\frac{n-1}{2}}\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right)+4\eta_e\sum_{\alpha=x,y,z}\left(\frac{\partial v_x}{\partial\alpha}+\frac{\partial v_\alpha}{\partial x}\right)\left(\frac{\partial v_z}{\partial\alpha}+\frac{\partial v_\alpha}{\partial z}\right);\\
&\tau_{yz}=KI_2^{\frac{n-1}{2}}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)+4\eta_e\sum_{\alpha=x,y,z}\left(\frac{\partial v_y}{\partial\alpha}+\frac{\partial v_\alpha}{\partial y}\right)\left(\frac{\partial v_z}{\partial\alpha}+\frac{\partial v_\alpha}{\partial z}\right).
\end{aligned}\right\}\qquad(11)$$

/60

Substituting the resultant expressions (11) in the equation of motion for a continuous medium (10), and performing certain transformations, we finally get

$$
\left.\begin{aligned}
&\varrho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\varrho g_x+K\Bigg[I_2^{\frac{n-1}{2}}\nabla^2 v_x+\\
&+2\frac{\partial\left(I_2^{\frac{n-1}{2}}\right)}{\partial x}\frac{\partial v_x}{\partial x}+\frac{\partial\left(I_2^{\frac{n-1}{2}}\right)}{\partial y}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)+\frac{\partial\left(I_2^{\frac{n-1}{2}}\right)}{\partial z}\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right)\Bigg]+\\
&+4\eta_e\Bigg[\frac{\partial}{\partial x}\sum_{\alpha=x,y,z}\frac{\partial v_x}{\partial\alpha}\cdot\frac{\partial v_\alpha}{\partial x}+\frac{\partial}{\partial y}\sum_{\alpha=x,y,z}\left(\frac{\partial v_x}{\partial\alpha}+\frac{\partial v_\alpha}{\partial x}\right)\left(\frac{\partial v_\alpha}{\partial y}+\frac{\partial v_y}{\partial\alpha}\right)+\\
&\qquad+\frac{\partial}{\partial z}\sum_{\alpha=x,y,z}\left(\frac{\partial v_x}{\partial\alpha}+\frac{\partial v_\alpha}{\partial x}\right)\left(\frac{\partial v_z}{\partial\alpha}+\frac{\partial v_\alpha}{\partial z}\right)\Bigg];\\
&\varrho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial p}{\partial y}+\varrho g_y+\\
&\quad+K\Bigg[I_2^{\frac{n-1}{2}}\nabla^2 v_y+\frac{\partial\left(I_2^{\frac{n-1}{2}}\right)}{\partial x}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)+2\frac{\partial\left(I_2^{\frac{n-1}{2}}\right)}{\partial y}\frac{\partial v_y}{\partial y}+\\
&+\frac{\partial\left(I_2^{\frac{n-1}{2}}\right)}{\partial z}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)\Bigg]+4\eta_e\Bigg[\frac{\partial}{\partial x}\sum_{\alpha=x,y,z}\left(\frac{\partial v_x}{\partial\alpha}+\frac{\partial v_\alpha}{\partial x}\right)\left(\frac{\partial v_\alpha}{\partial y}+\frac{\partial v_y}{\partial\alpha}\right)+\\
&\quad+\frac{\partial}{\partial y}\sum_{\alpha=x,y,z}\frac{\partial v_y}{\partial\alpha}\cdot\frac{\partial v_\alpha}{\partial y}+\frac{\partial}{\partial z}\sum_{\alpha=x,y,z}\left(\frac{\partial v_y}{\partial\alpha}+\frac{\partial v_\alpha}{\partial y}\right)\left(\frac{\partial v_z}{\partial\alpha}+\frac{\partial v_\alpha}{\partial z}\right)\Bigg];\\
&\varrho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\varrho g_z+\\
&\quad+K\Bigg[I_2^{\frac{n-1}{2}}\nabla^2 v_z+\frac{\partial\left(I_2^{\frac{n-1}{2}}\right)}{\partial x}\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right)+\frac{\partial\left(I_2^{\frac{n-1}{2}}\right)}{\partial y}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)+\\
&\quad+2\frac{\partial\left(I_2^{\frac{n-1}{2}}\right)}{\partial z}\frac{\partial v_z}{\partial z}\Bigg]+4\eta_e\Bigg[\frac{\partial}{\partial x}\sum_{\alpha=x,y,z}\left(\frac{\partial v_x}{\partial\alpha}+\frac{\partial v_\alpha}{\partial x}\right)\left(\frac{\partial v_z}{\partial\alpha}+\frac{\partial v_\alpha}{\partial z}\right)+\\
&\quad+\frac{\partial}{\partial y}\sum_{\alpha=x,y,z}\left(\frac{\partial v_y}{\partial\alpha}+\frac{\partial v_\alpha}{\partial y}\right)\left(\frac{\partial v_z}{\partial\alpha}+\frac{\partial v_\alpha}{\partial z}\right)+\frac{\partial}{\partial z}\sum_{\alpha=x,y,z}\frac{\partial v_z}{\partial\alpha}\cdot\frac{\partial v_\alpha}{\partial z}\Bigg].
\end{aligned}\right\}\qquad(12)
$$

It is often convenient to use an orthogonal curvilinear coordinate system such as a cylindrical system instead of the rectangular coordinate system. In this case we have the following relationship between the cylindrical coordinates r, ϕ, z and the Cartesian coordinates x, y, z: /61

$$
\left.\begin{aligned}
x&=r\cos\varphi; & \phi&=\arctan\frac{y}{x};\\
y&=r\sin\varphi; & r&=\sqrt{x^2+y^2};\\
&& z&=z.
\end{aligned}\right\}\qquad(13)
$$

The Lame coefficients take the following form:

$$H_r = \sqrt{\sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial q_k}\right)^2} = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = 1;$$

$$H_\varphi = r; \qquad h_z = 1. \tag{14}$$

The projections of body force F on the cylindrical coordinate axes will be ρg_r, ρg_ϕ, ρg_z. Then after computing all terms in the equations of system (12) from the corresponding formulas, we get an equation for the motion of a rheological fluid in cylindrical coordinates:

$$\begin{aligned}
&\varrho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\varphi}{r}\cdot\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \varrho g_r + \\
&+\frac{k}{r}\left[rI_2^{\frac{n-1}{2}}\nabla^2 v_r + 2\frac{\partial(rI_2^{\frac{n-1}{2}})}{\partial r}\cdot\frac{\partial v_r}{\partial r} + \frac{\partial(rI_2^{\frac{n-1}{2}})}{r\partial\varphi}\left(\frac{\partial v_r}{\partial\varphi} + \frac{\partial v_\varphi}{\partial r}\right) + \right.\\
&\left.+\frac{\partial(rI_2^{\frac{n-1}{2}})}{\partial z}\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) - 2I_2^{\frac{n-1}{2}}\frac{\partial v_\varphi}{\partial\varphi}\right] + \\
&+4\frac{\eta_e}{r}\left[\frac{\partial}{\partial r}\sum_{\alpha=x,y,z}\frac{\partial v_\alpha}{\partial r}\cdot\frac{\partial v_r}{\partial\alpha} + \frac{\partial}{\partial\varphi}\sum_{\alpha=x,y,z}\left(\frac{\partial v_r}{\partial\alpha} + \frac{\partial v_\alpha}{\partial r}\right)\times\right.\\
&\times\left(\frac{\partial v_\varphi}{\partial\alpha} + \frac{\partial v_\alpha}{\partial\varphi}\right) + r\frac{\partial}{\partial z}\sum_{\alpha=x,y,z}\left(\frac{\partial v_z}{\partial\alpha} + \frac{\partial v_\alpha}{\partial z}\right)\left(\frac{\partial v_r}{\partial\alpha} + \frac{\partial v_\alpha}{\partial r}\right) - \\
&\left.-\sum_{\alpha=x,y,z}\frac{\partial v_\alpha}{\partial\varphi}\cdot\frac{\partial v_\varphi}{\partial\alpha}\right].
\end{aligned} \tag{15}$$

If we assume that $K = \mu = \text{const}$, $n = 1$ and $\eta_e = 0$, systems of equations (12) and (13) take on the form of Navier-Stokes equations for a viscous fluid.

Equations (12) and (15) may be used for studying laminar motion of non-Newtonian fluids with regard to the effect of the transverse viscosity, i.e. elasticity. The generalized Reynolds number which characterizes the nature of flow in the form considered by Metzner [4] /62

$$\mathrm{Re} = \frac{vd}{\nu} = \frac{\varrho v^{?} d}{K\left(\frac{8v}{d}\right)^{n-1}} = \mathrm{Re}_{\text{Metzn}}. \tag{16}$$

does not account for the effect of elasticity and therefore should be made more precise for elastic flows. Studies conducted by Harries [5] showed that the Reynolds number for flows which have the transverse viscosity effect is a little lower than the values calculated by formula (16):

$$Re_{el} = \frac{Re_{Metzn}}{c} \tag{17}$$

Here, c is the correction coefficient which takes account of elasticity of the flow.

$$c = \left[1 + \frac{(P_{nn} - P_{ee})^2}{(2\tau)^2}\right]^{\frac{1}{2}},$$

where $P_{nn} - P_{ee}$ is the difference in the tangential components of flow pressure (n and e are the axial and radial directions).

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RIEMANN AND ALFVEN WAVES

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ABSTRACT: The authors consider equations of magnetohydrodynamics with conduction anisotropy. The conditions on the discontinuity surface are written out as well as expressions for the force and moment acting on a body in a flow. Waves of finite amplitude are studied for the symmetric problem.

A system of equations in magnetohydrodynamics with conduction anisotropy [10] with regard to ohmic dissipation takes the following form: /63

$$\operatorname{div} \vec{H} = 0; \tag{1}$$

$$\frac{\partial \vec{H}}{\partial t} = \operatorname{rot} (\vec{U} \times \vec{H} + \beta V \vec{H} \times \operatorname{rot} \vec{H} - \nu_m \operatorname{rot} \vec{H}); \tag{2}$$

$$\frac{\partial}{\partial t} \cdot \frac{1}{V} + \operatorname{div} \frac{\vec{U}}{V} = 0; \tag{3}$$

$$\frac{\partial \vec{U}}{\partial t} + \nabla \frac{\vec{U}^2}{2} - \vec{U} \times \operatorname{rot} \vec{U} = -V \left(\operatorname{grad} p + \frac{\vec{H} \times \operatorname{rot} \vec{H}}{4\pi} \right); \tag{4}$$

$$\frac{\partial}{\partial t} \left(\frac{\vec{U}^2}{2V} + \frac{\varepsilon}{V} + \frac{H^2}{8\pi} \right) + \operatorname{div} \vec{q} = 0; \tag{5}$$

$$\vec{q} = \left(\frac{\vec{U}^2}{2V} + \frac{\varepsilon}{V} + p \right) \vec{U} + \frac{\vec{H} \times (\vec{U} \times \vec{H})}{4\pi} - \frac{\nu_m}{4\pi} \vec{H} \times \operatorname{rot} \vec{H}; \tag{6}$$

$$\vec{E} = \frac{1}{c} [\vec{H} \times (\vec{U} - \beta V \operatorname{rot} \vec{H}) + \nu_m \operatorname{rot} \vec{H}], \quad \beta = \frac{cM}{4\pi e}; \tag{7}$$

$$\vec{j} = \frac{c}{4\pi} \operatorname{rot} \vec{H}, \tag{8}$$

where $\vec{H}$ and $\vec{E}$ are the magnetic and electric fields, respectively; $\vec{U}$ is velocity; $\vec{q}$ is the energy current density vector; V is specific volume; p is pressure; ε is the internal energy of a unit mass of the fluid; c is the velocity of light; e is the constant of electronic charge; M is the fluid mass per /64
electron; ν_m is the coefficient of magnetic viscosity.

In the case of an ideal gas ($\nu_m = 0$) with certain assumptions [3] expressions (5) and (6) may be used to give

$$p = p(V, S); \quad \frac{\partial}{\partial t} \cdot \frac{S}{V} + \operatorname{div} \frac{S\vec{U}}{V} = 0, \tag{9}$$

where S is the entropy of a unit mass of the fluid.

Equations (1)-(4) and (9) forms a closed system. In the case of an incompressible fluid (V = const) it has been shown [3] that it is often sufficient to use only equations (1)-(4).

Let us consider a stationary discontinuity surface with unit normal $\vec{n}$. It is convenient to use the divergent form of the equations in setting up the conditions on this surface. Therefore, we also write out relationships (2), (4) and (8) in divergent form:

$$\frac{\partial H_i}{c\partial t} + \frac{\partial \varepsilon_{ikl} E_l}{\partial x_k} = 0, \quad \operatorname{div} \vec{j} = 0; \tag{10}$$

$$\frac{\partial}{\partial t} \cdot \frac{U_i}{V} + \frac{\partial \Pi_{ik}}{\partial x_k} = 0, \quad \Pi_{ik} = \frac{U_i U_k}{V} + \left(p + \frac{\vec{H}^2}{8\pi}\right)\delta_{ik} - $$
$$- \frac{1}{4\pi} H_i H_k \quad (i, k, l = 1, 2, 3), \tag{11}$$

where ε_{ikl} is a unit antisymmetric tensor of third order; Π_{ik} is the pulse flux density tensor.

Equations (1), (3), (5), (8), (10) and (11) give us the following conditions on this surface:

$$[\mu H_n] = [\vec{n} \times \vec{E}] = [(\operatorname{rot} \vec{H})_n] = \left[\frac{U_n}{V}\right] = [\Pi_{in}] = [q_n] = 0, \tag{12}$$

where the subscript n designates the components of the vectors and tensors which are normal to the surface; μ is magnetic permeability.

Let us consider a fixed solid located in the flow. Using the conditions of continuity for the normal components of the pulse flux tensor on the surface of the solid, we get the following expressions for the force $\vec{Q}$ and the moment $\vec{M}$ acting on the solid:

$$\vec{Q} = \oint \vec{f} dS \quad \vec{M} = \oint \vec{r} \times \vec{f} dS; \tag{13}$$

$$\vec{f} = -\left(p + \frac{\vec{H}^2}{8\pi}\right)\vec{n} + \frac{H_n}{4\pi}\vec{H} - \frac{U_n}{V}\vec{U}, \tag{14}$$

where dS is an element of area on the surface of the solid.

Let us consider a problem in which the quantities $\vec{H}$, $\vec{U}$ and V are independent of the coordinates x_2 and x_3 with additional conditions which will be defined during the course of the discussion. We shall call this the symmetric problem for the sake of brevity. /65

Let us consider the symmetric problem for the case of an ideal gas ($\nu_m = 0$). From equation (1) and the first component of equation (2), we get $H_1 = H_0 = \text{const}$ (the subscript 1 indicates the x-components of the vectors where $x \equiv x_1$). Let $H_0 = 0$. Then the term appearing in equation (2) which is proportional to β will be equal to 0. Consequently the magnetic lines of force are frozen in the material.

In solving the equation of continuity, we introduce the particle function ψ:

$$\frac{1}{V} = \frac{\partial\psi}{\partial x}; \quad v = -\frac{\partial\psi/\partial t}{\partial\psi/\partial x}; \quad v = U_1. \tag{15}$$

Using relationships (15) and assuming that the entropy S is also independent of coordinates x_2 and x_3, we write the general solutions of the second and third components of equations (2), (4) and (9) in the form

$$\vec{H} = \vec{\varkappa}(\psi)/V, \quad \vec{\varkappa} = (\varkappa_2, \varkappa_3);$$
$$\vec{u} = \vec{u}(\psi), \quad \vec{u} = (U_2, U_3); \quad S = S(\psi). \tag{16}$$

Here $\vec{\kappa}(\psi)$, $\vec{u}(\psi)$, and $S(\psi)$ are arbitrary functions of their argument. Substituting equations (16) in formula (7), we get an expression for the electric field

$$\vec{E} = \frac{1}{c}\left\{\frac{v}{V}\vec{\varkappa}\times\vec{e} + \left[\frac{\vec{c}(\vec{\varkappa}\times\vec{u})}{V} - \beta\vec{\varkappa}\frac{\partial}{\partial x}\cdot\frac{\vec{\varkappa}}{V}\right]\vec{e}\right\}, \tag{17}$$

where $\vec{e}$ is the unit vector parallel to the x-axis.

Substituting relationships (16) for $\vec{H}$ and S in the first component of equation of motion (4), we convert to Lagrange variables (ψ, t) in the resultant relationship. By converting to Lagrange variables in expressions (15), we eliminate the x-coordinate from the resultant equations. In this way we get

$$\frac{\partial v}{\partial t} = -\frac{\partial P}{\partial \psi}, \quad \frac{\partial v}{\partial \psi} = \frac{\partial V}{\partial t}; \quad P = p(V, \psi) + \frac{\vec{\varkappa}^2(\psi)}{8\pi V^2}. \tag{18}$$

Kaplan and Stanyukovich made certain assumptions and reduced the symmetric problem in non-dissipative magnetogasdynamics with isotropic conductivity to a gas dynamic problem with altered equation of state [2]. The result was later confirmed in [4] during investigations in magnetogasdynamics and collisionless magnetogasdynamics with anisotropic pressure with several more general assumptions. As shown above, the symmetric problem of nondissipative magnetogasdynamics with anisotropic conductivity also reduces to a gas dynamics /66
problem with altered equation of state. This situation means that most of the known results in ordinary gas dynamics may be extended to the case under consideration. In particular, Riemann waves may be easily studied [2, 4, 5].

Let us consider the symmetric problem of a fluid with anisotropic conductivity (V = const). In this case we get the following integrals of symmetry from equations (1) and (3) and the first components of equations (2) and (4):

$$\begin{gathered} H_1 = H_0; \quad U_1 = U_0; \\ U_0'x + V\left(p + \frac{H_2^2 + H_3^2}{8\pi}\right) - Q_2x_2 - Q_3x_3 = f(t). \end{gathered} \tag{19}$$

Here, H_0 is an arbitrary constant; $U_0(t)$, $Q_2(t)$, $Q_3(t)$ and $f(t)$ are arbitrary functions of time. Let us note that integrals of symmetry (19) coincide with those given in [9] where the case of a fluid with isotropic conductivity was considered.

Using the integrals of symmetry, we convert the second and third components of equations (2) and (4) to the following system of linear equations

$$\left.\begin{gathered}\left(\frac{d}{dt}-\nu_m\frac{\partial^2}{\partial x^2}\right)\vec{h}=H_0\frac{\partial}{\partial x}\left(\vec{u}-\beta V\vec{e}\times\frac{\partial\vec{h}}{\partial x}\right);\\ \frac{d\vec{u}}{dt}=\frac{VH_0}{4\pi}\cdot\frac{\partial\vec{h}}{\partial x}-\vec{Q};\\ \vec{h}=(H_2,H_3);\quad \vec{u}=(U_2,U_3);\quad \vec{Q}=(Q_2,Q_3);\\ \frac{d}{dt}=\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x}.\end{gathered}\right| \tag{20}$$

In view of the linearity of equations (20), fields $\vec{h}$ and $\vec{u}$ satisfy the principle of superposition [9, 10]. Because of the linearity of system (20), the well-known methods of mathematical physics may be used in solving the problems. After determining the fields $\vec{h}$ and $\vec{u}$ from equations (20) and substituting these values in the third relationship of system (19), the pressure p may be determined. Let us point out that a number of problems in dissipative magnetohydrodynamics with anisotropic conductivity have been studied on the basis of systems of type (20) in [6-8, 11 and others].

When $\nu_m = H_0 = 0$, the general solution of (20) takes the form

$$\vec{u}=\vec{\varphi}(\psi)-\int\vec{Q}dt;\quad \vec{h}=\vec{h}(\psi);\quad \psi=x-\int U_0dt. \tag{21}$$

where $\vec{\phi}$ and $\vec{h}$ are arbitrary functions of their argument.

Let $H_0 \neq 0$. By introducing some potential, we solve the equation of electrical induction (20): /67

$$\begin{gathered}\vec{h}=\vec{e}\times\frac{\partial\vec{a}}{\partial x};\quad H_0\vec{u}=\left(\frac{d}{dt}-\nu_m\frac{\partial^2}{\partial x^2}\right)(\vec{e}\times\vec{a})-\beta VH_0\frac{\partial^2\vec{a}}{\partial x^2};\\ \vec{A}=\vec{a}(x,t)+\frac{H_0}{2}\vec{e}\times\vec{q};\quad \vec{a}=(a_2,a_3);\quad \vec{q}=(x_2,x_3)\end{gathered} \tag{22}$$

where $\vec{A}$ is the vector potential. To determine the quantity $\vec{a}$ we substitute expression (22) in equation of motion (20):

$$\left[\frac{d}{dt}\left(\frac{d}{dt}-\nu_m\frac{\partial^2}{\partial x^2}\right)-\frac{VH_0^2}{4\pi}\cdot\frac{\partial^2}{\partial x^2}\right]\vec{a}+\beta VH_0\frac{d}{dt}\cdot\frac{\partial^2}{\partial x^2}(\vec{e}\times\vec{a})=H_0\vec{e}\times\vec{Q}. \tag{23}$$

Let $\vec{Q} = 0$. Solving the first equation in system (23) and substituting the result in the second equation, we get

$$a_2=\beta VH_0\frac{d}{dt}\cdot\frac{\partial^2 F}{\partial x^2}; \tag{24}$$

$$a_3=\left[\frac{d}{dt}\left(\frac{d}{dt}-\nu_m\frac{\partial^2}{\partial x^2}\right)-\frac{VH_0^2}{4\pi}\cdot\frac{\partial^2}{\partial x^2}\right]F;$$

$$\left\{\left[\frac{d}{dt}\left(\frac{d}{dt}-\nu_m\frac{\partial^2}{\partial x^2}\right)-\frac{VH_0^2}{4\pi}\cdot\frac{\partial^2}{\partial x^2}\right]^2+(\beta VH_0)^2\left(\frac{d}{dt}\cdot\frac{\partial^2}{\partial x^2}\right)^2\right\}F=0. \tag{25}$$

Thus, the problem reduces to the single linear equation (25). We determine all physical quantities from formulas (19), (22) and (24). Assuming that $\nu_m = 0$, in expressions (20), (22)-(25), we get the relationships for the symmetric problem of nondissipative magnetohydrodynamics with anisotropic conductivity.

Let us consider the case of propagation of a wave of finite amplitude in a nondissipative ($\nu_m = 0$) fluid along a constant field H_0. Using expression (25) and assuming that $U_0 = 0$, $F \approx \exp i(kx - \omega t)$, we arrive at the dispersion equation

$$(\omega^2-k^2v_\alpha^2)^2-(\beta VH_0\omega k^2)^2=0;\quad v_\alpha^2=\frac{VH_0^2}{4\pi}, \tag{26}$$

where v_α is the Alfven velocity.

Finding the phase velocity $v_\phi = \omega/k$ from equations (26), we get

$$v_\phi=(\sqrt{1+\varkappa^2}+\varkappa)v_\alpha;\quad v_\phi=\frac{v_\alpha}{\sqrt{1+\varkappa^2}+\varkappa};\quad \varkappa=\frac{kc}{2\Omega};\quad \Omega^2=\frac{4\pi ne^2}{M};\quad nM=\frac{1}{V}. \tag{27}$$

Here n is the number of electrons per unit of volume. Relationships (27) describe a fast-travelling ($v_\Phi > v_\alpha$) or high-frequency wave, and a slow-travelling ($v_\Phi < v_\alpha$) or low-frequency wave. The greater the parameter κ (i.e. the shorter the wave length and the lower the density), the greater will be the difference between the phase velocities of these waves and the Alfven velocity. When $\kappa \ll 1$ (i.e. when the waves are fiarly long and the densities are fairly high), the phase velocities of both waves are close to the Alfven velocity. /68

Let us consider the problem of a simple resonator which consists of a plasma layer confined by metallic surfaces at $x = 0$ and $x = x_0$. The magnetic field H_0 is perpendicular to these surfaces. By using a solution of the form $B = B_0 \times \sin kx \sin \omega t$ to satisfy boundary conditions $E_2 = E_3 = 0$ when $x = 0$ and $x = x_0$, we arrive at dispersion equation (26). In this case, $k = s\pi/x_0$ ($s = 1, 2, 3, \ldots$). After transformations, we get for high-frequency and low-frequency standing waves:

$$\omega = \frac{1+\sqrt{1+2N}}{N}\,\omega_H; \quad \omega = \frac{2\omega_H}{1+\sqrt{1+2N}}; \qquad N \equiv \frac{n8e^2x_0^2}{\pi Mc^2s^2}; \quad \omega_H \equiv \frac{eH_0}{Mc}. \tag{28}$$

Let us consider one the modes. As implied by equation (28), the frequencies of each wave increase linearly with an increase in the field H_0 at a fixed value of the dimensionless density N. The frequencies of both waves decrease when N is increased while the magnetic field is held constant. In this case, if $N \ll 1$, the frequency of the low-frequency wave is closer to ω_H. Dispersion relationships (27) and (28) may be derived from dispersion relationships for the extraordinary and ordinary waves [1] assuming that

$$\left(\frac{\omega}{kc}\right)^2 \ll 1; \left(\frac{m}{M}\right)\left(\frac{kc}{\Omega}\right)^2 \ll 1,$$

where m is the mass of the electron.

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CRITERION FOR ENGINEERING STABILITY OF WING MOTION CLOSE TO THE INTERFACE BETWEEN TWO MEDIA

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ABSTRACT: Differential equations are derived for non-linear vibrations of a wing under flight conditions close to a deflector, the conditions for motion stability of the wing in the vertical plane are considered and a method is proposed for finding the region of asymptotic stability for vibrations of this type.

Engineering Stability for Motion of a Wing in the Case of Perturbations With Respect to Height

Let us consider the non-linear vibrations of a wing in the vertical plane when the wing is moving with a constant horizontal velocity component v close to the interface between two media. We assume that the angle between the chord of the wing and the interface does not change with time. The coordinate system xoy is as shown in Figure 1. /69

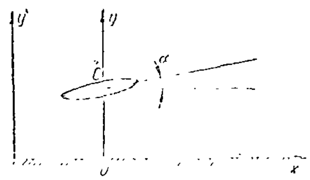

Figure 1. Diagram of Wing Motion.

The oscillations of point C located at the center of gravity of the wing are described by the equation

$$m\ddot{y} = -mg + C_y(\lambda, \alpha, y) S \frac{\rho v^2}{2} - C_y'(\lambda, \alpha, y) S \frac{\rho v^2}{2} \dot{y} \quad (1)$$

where m, λ and S are the mass, aspect ratio and area of the wing respectively; ρ is the mass density of the air; α is the instantaneous angle of attack equal to y/v;

$$C_y'(\lambda, \alpha, y) = \left. \frac{\partial C_y(\lambda, \varphi, y)}{\partial \varphi} \right|_{\varphi = \alpha},$$

$C_y(\lambda, \phi, y)$ is the coefficient of lift of the wing as a function of aspect ratio λ, angle of attack ϕ and height above the interface y.

Due to the effect of the interface, the relationship between coefficients $C_y(\lambda, \alpha, y)$ and $C_y'(\lambda, \alpha, y)$ on the one hand and the values of λ and y on the other is non-linear [2]. Two methods may be used for determining this analytical relationship. The first method is as follows. If experimental data on

C_y and C_y' as functions of λ and y are accumulated in the form of tables or graphs, these relationships may be approximated by the method of least squares. Let us assume that the given relationships take the form /70

$$\begin{aligned} C_y(\lambda, \alpha, y) &= (a_{00} + a_{01}\lambda + a_{02}\lambda^2) + (a_{10} + a_{11}\lambda + a_{12}\lambda^2)\, y + \\ &+ (a_{20} + a_{21}\lambda + a_{22}\lambda^2)\, y^2; \\ C_y'(\lambda, \alpha, y) &= (b_{00} + b_{01}\lambda + b_{02}\lambda^2) + (b_{10} + b_{11}\lambda + b_{12}\lambda^2)\, y + \\ &+ (b_{20} + b_{21}\lambda + b_{22}\lambda^2)\, y^2, \end{aligned} \tag{2}$$

where a_{ij}, b_{ij} $(i, j = 0, 1, 2)$ are specific quantities corresponding to a predetermined value of α determined during analysis of experimental data by the method of least squares.

The second method is to find the analytical relationships by theoretical hydromechanical investigation.

Taking consideration of equation (1), we determine the height h_0 at which the wing will be in equilibrium from the expression

$$\begin{aligned} a_{00} + a_{01}\lambda + a_{02}\lambda^2 + (a_{10} + a_{11}\lambda + a_{12}\lambda^2)\, h_0 + (a_{20} + a_{21}\lambda + \\ + a_{22}\lambda^2)\, h_0^2 = \frac{2mg}{S\rho v^2}. \end{aligned} \tag{3}$$

Solving equation (3), we get

$$h_0 = H_0(\lambda, m, S, v) \tag{4}$$

The equation for disturbed motion of the wing may then be written in the form

$$\begin{aligned} m\Delta\ddot{y} &= \frac{\partial C_y(\lambda, \alpha, y)}{\partial y}\bigg|_{y=H_0(\lambda,m,S,v)} S\frac{\rho v^2}{2}\Delta y + \\ &+ \frac{1}{2}\cdot\frac{\partial^2 C_y(\lambda, \alpha, y)}{\partial y^2}\bigg|_{y=H_0(\lambda,m,S,v)} S\frac{\rho v^2}{2}(\Delta y)^2 - \\ &- C_y'(\lambda, \alpha, H_0)\, S\frac{\rho v^2}{2}\Delta\dot{y} - \frac{\partial C_y'(\lambda, \alpha, y)}{\partial y}\bigg|_{y=H_0(\lambda,m,S,v)} S\frac{\rho v^2}{2}\Delta y\Delta\dot{y} - \\ &- \frac{1}{2}\cdot\frac{\partial^2 C_y'(\lambda, m, S, v)}{\partial y^2}\bigg|_{y=H_0(\lambda,m,S,v)} S\frac{\rho v}{2}(\Delta y)^2\Delta\dot{y}. \end{aligned} \tag{5}$$

Let us introduce the notation /71

$$
\left.
\begin{aligned}
\Phi_1(\lambda, m, S, v) &= -\left.\frac{\partial C_y(\lambda, \alpha, y)}{\partial y}\right|_{y=H_0(\lambda, m, S, v)} S\frac{\varrho v^2}{2m};\\
\Phi_2(\lambda, m, S, v) &= \frac{1}{2}\cdot\left.\frac{\partial^2 C_y(\lambda, \alpha, y)}{\partial y^2}\right|_{y=H_0(\lambda, m, S, v)} S\frac{\varrho v^2}{2m};\\
\Phi_3(\lambda, m, S, v) &= C'_y(\lambda, \alpha, H_0(\lambda, m, S, v))\, S\frac{\varrho v^2}{2m};\\
\Phi_4(\lambda, m, S, v) &= -\left.\frac{\partial C_y(\lambda, \alpha, y)}{\partial y}\right|_{y=H_0(\lambda, m, S, v)} S\frac{\varrho v}{2m};\\
\Phi_5(\lambda, m, S, v) &= -\frac{1}{2}\cdot\left.\frac{\partial^2 C'_y(\lambda, \alpha, y)}{\partial y^2}\right|_{y=H_0(\lambda, m, S, v)} S\frac{\varrho v}{2m}.
\end{aligned}
\right\}
\tag{6}
$$

Equation (5) is rewritten with regard to notation (6):

$$
\Delta\ddot{y} = -\Phi_1 \Delta y + \Phi_2 (\Delta y)^2 - \Phi_3 \Delta\dot{y} + \Phi_4 \Delta y \Delta\dot{y} + \Phi_5 (\Delta y)^2 \Delta\dot{y}. \tag{7}
$$

Since C_y decreases with an increase in height y and increases with an increase in the angle of attack [1], it follows that

$$
\Phi_1(\lambda, m, S, v) > 0; \qquad \Phi_3(\lambda, m, S, v) > 0, \tag{8}
$$

i.e. the motion of the wing is asymptotically stable in the linear approximation.

However, the singularities involved in the motion of a structure close to a deflector make it necessary for us to investigate problems in what is called engineering stability rather than Lyapunov stability in the classical sense. In other words, we must select the parameters of the system and the permissible region of initial perturbations in such a way that the solution of equation (7) does not exceed the level

$$
\Delta y = \pm H_0(\lambda, m, S, v) \tag{9}
$$

assuming fulfillment of the condition

$$
\lim_{t\to\infty} \Delta y(t) = 0,
$$

i.e. it is necessary to establish under what condition the wing in its motion will not touch the interface between the two media or deviate from this interface by a height greater than $2H_0(\lambda, m, S, v)$ and will be asymptotically stable in motion.

For this purpose we use N. G. Chetayev's method [3] to construct a Lyapunov function in the form

$$v(\Delta y, \Delta\dot{y}) = \frac{\Phi_3 + \Phi_1^2 + \Phi_1}{2\Phi_3\Phi_1}(\Delta y)^2 + \frac{1}{\Phi_1}\Delta y\Delta\dot{y} + \frac{\Phi_1 + 1}{2\Phi_3\Phi_1}(\Delta\dot{y})^2. \tag{10}$$

Then the function /72

$$\left(\frac{dv}{dt}\right)_7 = (\Delta y)^2\left[-1 + \frac{\Phi_2}{\Phi_1}\Delta y + \frac{\Phi_3\Phi_4 + (\Phi_1 + 1)\Phi_2}{\Phi_3\Phi_1}\Delta\dot{y} + \frac{\Phi_5}{\Phi_1}\Delta y\Delta\dot{y}\right] + (\Delta\dot{y})^2\left[-1 + \frac{\Phi_1 + 1}{\Phi_3\Phi_1}\Phi_4\Delta y + \frac{(\Phi_1 + 1)\Phi_5}{\Phi_3\Phi_1}(\Delta y)^2\right] \tag{11}$$

will be negatively defined in the region

$$|\Delta y| < H_0(\lambda, m, S, v),$$

$$|\Delta\dot{y}| < y_1, \tag{12}$$

if

$$\begin{cases} -1 + \left|\frac{\Phi_1 + 1}{\Phi_3\Phi_1}\Phi_4\right| H_0(\lambda, m, S, v) + \frac{(\Phi_1 + 1)\Phi_5}{\Phi_3\Phi_1} H_0^2(\lambda, m, S, v) < 0 \\ -1 + \left|\frac{\Phi_2}{\Phi_1}\right| H_0(\lambda, m, S, v) < 0. \end{cases} \tag{13}$$

In this case

$$\bar{y}_1 = \frac{1 - \left|\frac{\Phi_2}{\Phi_1}\right| H_0(\lambda, m, S, v)}{\left|\frac{\Phi_3\Phi_4 + (\Phi_1 + 1)\Phi_2}{\Phi_3\Phi_1}\right| + \left|\frac{\Phi_5}{\Phi_1}\right| H_0(\lambda, m, S, v)}. \tag{14}$$

Perturbation of the initial values in the region

$$\frac{\Phi_3 + \Phi_1^2 + \Phi_1}{2\Phi_3\Phi_1}(\Delta y(0))^2 + \frac{1}{\Phi_1}\Delta y(0)\Delta\dot{y}(0) + \frac{\Phi_1 + 1}{2\Phi_3\Phi_1}(\Delta\dot{y}(0))^2 < M, \tag{15}$$

where

$$M = \min\begin{cases} 2\dfrac{H_0^2(\lambda, m, S, v)\,\Phi_3\Phi_1}{\Phi_1+1}; \\ 2\dfrac{\left[1-\left|\dfrac{\Phi_2}{\Phi_1}\right| H_0(\lambda, m, S, v)\right]\Phi_3\Phi_1}{\left[\left|\dfrac{\Phi_3\Phi_4+(\Phi_1+1)}{\Phi_3\Phi_4}\right|+\left|\dfrac{\Phi_5}{\Phi_1}\right| H_0(\lambda, m, S, v)\right][\Phi_3+\Phi_1^2+\Phi_1]} \end{cases}$$

does not disrupt the stability of the system if the parameters of the wing satisfy conditions (13). This is because the function $v(\Delta y, \Delta\dot{y})$ decreases monotonically along the trajectories of the system in rectangle (12) in view of its construction, but since this function is a positively defined quadratic form,

$$\Delta y(t) \underset{t\to\infty}{\to} 0;$$

$$\Delta\dot{y}(t) \underset{t\to\infty}{\to} 0;$$

$$|\Delta y(t)| < H_0(\lambda, m, S, v),$$

i.e. condition (9) is fulfilled as was necessary. /73

Let us consider the equation for vibrations of a wing without regard to change in the angle of attack:

$$m\ddot{y} = -mg + C_y\left(\lambda, \alpha - \frac{\dot{y}}{v}, y\right) S\frac{\rho v^2}{2}. \tag{16}$$

By using the methods of Andronov's qualitative vibration theory, we may determine the condition where the solution of expression (16) without attenuation does not reach the level

$$\Delta y = -h_0,$$

i.e. the wing is not located on the interface between two media.

Without regard to attenuation, we write equation (5) in the form

$$\Delta\ddot{y} = a_1\Delta y + a_2\Delta y^2, \tag{17}$$

where the expressions

$$a_1 = \frac{\partial C_y(\lambda, \alpha, y)}{\partial y}\Big|_{y=h_0} S\frac{\varrho v^2}{2m} = [a_{10} + a_{11}\lambda + a_{12}\lambda^2 + 2h_0(a_{20} + a_{21}\lambda + a_{22}\lambda^2)] S\frac{\varrho v^2}{2};$$
$$a_2 = \frac{1}{2}\cdot\frac{\partial^2 C_y(\lambda, \alpha, y)}{\partial y^2}\Big|_{y=h_0} S\frac{\varrho v^2}{2m} = (a_{20} + a_{21}\lambda + a_{22}\lambda^2) S\frac{\varrho v^2}{2m} \tag{18}$$

satisfy equation (3) with regard to relationship (2).

We write formula (17) in the form /74

$$\frac{(\Delta\dot{y})^2}{2} = \frac{a_1(\Delta y)^2}{2} + \frac{a_2(\Delta y)^3}{3} + C \tag{19}$$

or

$$\Delta\dot{y} = \pm\sqrt{a_1(\Delta y_1)^2 + \frac{2}{3}a_2(\Delta y)^3 + 2C} \tag{20}$$

where C is the constant of integration.

In the phase plane Δy, $\Delta\dot{y}$ we get the qualitative picture shown in Figure 2, a and b.

If perturbation of the initial values $\Delta y(0)$, $\Delta\dot{y}(0)$ at the initial instant satisfies the condition

$$|\Delta\dot{y}(0)| \leqslant \sqrt{a_1(\Delta y(0))^2 + \frac{2}{3}a_2(\Delta y(0))^3 + \frac{2}{3}a_2h_0^3 + a_1h_0^2},$$

then

$$|\Delta y(t)| \leqslant h_0 \quad \text{when} \quad t > 0.$$

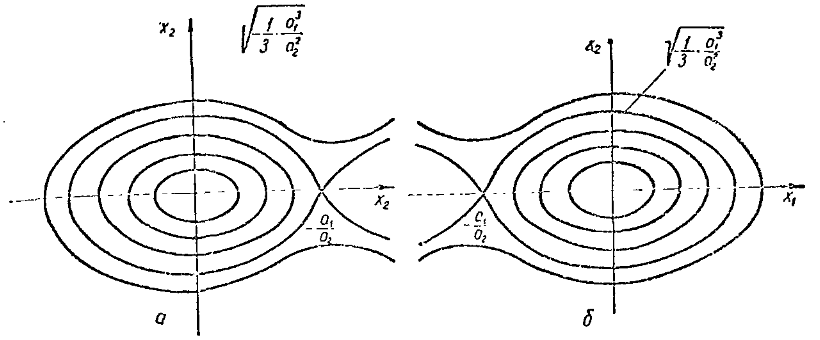

Figure 2. Diagram of zone of stable wing motion: a, for $a_2 > 0$; b, for $a_2 < 0$.

Parameters for Engineering Stability of a Wing in the Permissible Region of Perturbations With Respect to Height and Angle of Attack

The equations of disturbed motion of a wing with regard to the effect of the change in angle in inclination of the chord of the wing to the Qx-axis (see Figure 1) are as follows:

$$
\left.\begin{aligned}
m\Delta\ddot{y} &= \frac{\partial C_y(\lambda,\varphi_p,h_0)}{\partial y} S\frac{\rho v^2}{2}\Delta y + \frac{1}{2}\cdot\frac{\partial^2 C_y(\lambda,\varphi_p,h_0)}{\partial y^2} S\frac{\rho v^2}{2}\Delta y^2 + \\
&+ C'_y(\lambda,\varphi_p,h_0) S\frac{\rho v^2}{2}\Delta\varphi + \frac{\partial}{\partial y}C'_y(\lambda,\varphi_p,h_0) S\frac{\rho v^2}{2}\Delta y\Delta\varphi + \\
&+ \frac{1}{2}\cdot\frac{\partial^2}{\partial y^2}C'_y(\lambda,\varphi_p,h_0) S\frac{\rho v^2}{2}\Delta y^2\Delta\varphi - k_1\Delta\dot{y};\\
I\Delta\ddot{\varphi} &= m'_z(\lambda,\varphi_p,h_0)\, bS\frac{\rho v^2}{2}\Delta\varphi + \frac{\partial}{\partial y}m_z(\lambda,\varphi_p,h_0)\, bS\frac{\rho v^2}{2}\Delta y + \\
&+ \frac{1}{2}\cdot\frac{\partial^2}{\partial y^2}m_z(\lambda,\varphi_p,h_0)\, bS\frac{\rho v^2}{2}\Delta y^2 + \frac{\partial}{\partial y}m'_z(\lambda,\varphi_p,h_0)\, bS\frac{\rho v^2}{2}\Delta y\Delta\varphi + \\
&+ \frac{1}{2}\cdot\frac{\partial^2}{\partial y^2}m'_z(\lambda,\varphi_p,h_0)\, bS\frac{\rho v^2}{2}\Delta y^2\Delta\varphi - k_2\Delta\dot{\varphi},
\end{aligned}\right\} \quad (21)
$$

where b is the chord of the wing; k_1, k_2 are the damping coefficients of the medium; ϕ_p is some angle of attack which may be taken as equal in magnitude to the angle of attack for steady-state (undisturbed) motion.

Introducing the corresponding notation, we rewrite system (21) in the form /75

$$
\begin{aligned}
\Delta\ddot{y} &= \Delta y A_1 + \Delta\varphi A_2 + \Delta y^2 A_3 + \Delta y\Delta\varphi A_4 + \Delta y^2\Delta\varphi A_5 - \frac{k_1}{m}\Delta\dot{y};\\
\Delta\ddot{y} &= \Delta\varphi B_1 + \Delta y B_2 + \Delta y^2 B_3 + \Delta y\Delta\varphi B_4 + \Delta y^2\Delta\varphi B_5 - \frac{k_2}{I}\Delta\dot{\varphi}.
\end{aligned} \quad (22)
$$

Using N. G. Chetayev's method [3], we construct two Lyapunov functions for system (22):

$$\begin{aligned} v_1(\Delta y, \Delta\varphi) &= c_{11}\Delta y^2 + c_{12}\Delta y\Delta\varphi + c_{22}\Delta\varphi^2; \\ v_2(\Delta y, \Delta\varphi) &= d_{11}\Delta y^2 + d_{12}\Delta y\Delta\varphi + d_{22}\Delta\varphi^2. \end{aligned} \tag{23}$$

Then in order to satisfy conditions

$$\begin{aligned} |\Delta y(t)| &\leqslant h_0; \\ |\Delta\varphi(t)| &\leqslant \bar{\varphi} \end{aligned} \tag{24}$$

where t > 0, it is sufficient to fulfill the inequalities

$$\begin{aligned} &|A_2\Delta\varphi + A_3\Delta y^2 + A_4\Delta y\Delta\varphi + A_5\Delta y^2\Delta\varphi| \leqslant \\ &\leqslant \frac{h_0(4c_{11}c_{22} - c_{12}^2)}{\left(c_{11} + c_{22} + \frac{1}{2}c_{12}\right)4c_{22}(2\sqrt{c_{11}c_{22}} + c_{12})}; \\ &|B_2\Delta y + B_3\Delta y^2 + B_4\Delta y\Delta\varphi + B_5\Delta y^2\Delta\varphi| \leqslant \\ &\leqslant \frac{\bar{\varphi}(4d_{11}d_{22} - d_{12}^2)}{\left(d_{11} + d_{22} + \frac{1}{2}d_{12}\right)4d_{22}(2\sqrt{d_{11}d_{22}} + d_{12})}, \end{aligned} \tag{25}$$

in set (24), where

$$\begin{aligned} c_{11} &= \frac{\left(\frac{k_1}{m}\right)^2 + A_1^2 + A_1}{2\frac{k_1}{m}A_1}; \quad c_{12} = \frac{1}{A_1}; \quad c_{22} = \frac{A_1 + 1}{2\frac{k_1}{m}A_1}; \\ d_{11} &= \frac{\left(\frac{k_2}{I}\right)^2 + B_1^2 + B_1}{2\frac{k_2}{I}B_1}; \quad d_{12} = \frac{1}{B_1}; \quad d_{22} = \frac{B_1 + 1}{2\frac{k_1}{I}B_1}. \end{aligned} \tag{26}$$

It is assumed in this case that the initial perturbations satisfy the inequalities /76

$$\begin{aligned} c_{11}y^2(0) + c_{12}y(0)y'(0) + c_{22}(y(0))^2 &\leqslant \frac{h_0^2(4c_{11}c_{22} - c_{12}^2)}{4c_{22}}; \\ d_{11}\varphi^2(0) + d_{12}\varphi(0)\varphi'(0) + d_{22}(\varphi'(0))^2 &\leqslant \frac{\bar{\varphi}^2(4d_{11}d_{22} - d_{12}^2)}{4d_{22}}. \end{aligned} \tag{27}$$

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METHOD OF CALCULATING NON-LINEAR PITCHING OF HYDROFOIL VESSELS

V. I. Kovolev

ABSTRACT: The author considers motion of a hydrofoil vessel against the waves under conditions where the waves are regular. Non-stationary hydrodynamic forces are determined on the basis of the solution found by A. N. Panchenkov. The method of finite differences is used for solving equations of motion with variable coefficients.

Let us consider the motion of a vessel with shallow hydrofoils on regular sinusoidal waves (motion against the waves). We shall assume that the amplitudes of waves and oscillations are so small that the squares of oscillation amplitudes may be disregarded. We shall also disregard interaction between the forward and aft hydrofoils. The problem of hitching of a hydrofoil vessel was solved in similar formulation by I. T. Yegorov and M. M. Bun'kov [1]. /77

The proposed numerical method differs from that mentioned in that it may be used to account for the non-linearity of hydrodynamic forces which arise on the hydrofoils. This non-linearity is fairly appreciable in the case of shallow hydrofoils even where the oscillations are relatively small.

The coefficient of lift C_y which determines the vertical forces supporting the vessel at a given height depends on the submersion of the hydrofoil h_i, the effective angle of attack α_i and the relative vertical velocity of the hydrofoil γ_i. For the case of motion on quiet water, the values of γ_i coincide with the tangent of the angle between the line tangent to the trajectory and the horizon.

Assuming that all deviations are determined in a coordinate system which is in horizontal motion with velocity v_0, the ox-axis is located in the plane of the undisturbed surface of the water and is parallel to the motion of the vessel, and the oy-axis is directed vertically upward and passes through the center of gravity of the vessel. These parameters may be expressed as follows:

$$\left.\begin{aligned} h_i &= h_{0i} - y - x_i\varphi + h_{bi};\\ \alpha_i &= \alpha_{0i} + \varphi - \frac{y' + x_i\varphi' - v_y}{v_i};\\ \gamma_i &= \frac{y' + x_i\varphi' - h'_{bi}}{v_i}. \end{aligned}\right\} \qquad (1)$$

Here h_{0i} is the distance of the i-th hydrofoil from the free surface /78
during steady-state motion on quiet water; α_{0i} is the effective angle of attack corresponding to these conditions, $\alpha_{0i} = \alpha_k + \alpha_0$ (α_k is the steady-state angle of attack, α_0 is the angle of zero lift for the hydrofoil); y, y', ϕ and ϕ' are the vertical and angular displacements and their derivatives with respect to time; displacements which reduce settling and increase the angle of attack of the hydrofoil are taken as positive; x_i is the distance along the horizontal from the center of gravity of the vessel to the center of pressure of the i-th hydrofoil; h_{bi} and h'_{bi} are the instantaneous values of displacement and velocity for the wave relief in the cross-section corresponding to the center of pressure of the i-th hydrofoil; $v_i = v(t) = v_0 - v_{xi}$ is the velocity of the i-th hydrofoil with respect to the water; v_x and v_y are the horizontal and vertical components of the orbital velocity of water particles at points corresponding to the position of the centers of pressure of the hydrofoils; v_0 is the velocity of motion of the vessel.

The lift increment may be represented in the form

$$\Delta P = \frac{\partial P}{\partial y} y + \frac{\partial P}{\partial y'} y' + \frac{\partial P}{\partial \varphi} \varphi + \frac{\partial P}{\partial \varphi'} \varphi' + \frac{\partial P}{\partial h_W} h_W + \frac{\partial P}{\partial h'_W} h'_W + \frac{\partial P}{\partial v_y} v_y + \frac{\partial P}{\partial v_x} v_x \tag{2}$$

or

$$\left.\begin{aligned}
\frac{\partial P_i}{\partial y} &= -\frac{\partial C_{yi}}{\partial h} \cdot \frac{\rho v_0^2}{2} S_i; \\
\frac{\partial P_i}{\partial y'} &= \frac{\rho v_0 S_i}{2} \left(-\frac{\partial C_{yi}}{\partial \alpha} + \frac{\partial C_{yi}}{\partial \gamma} \right); \\
\frac{\partial P_i}{\partial \varphi} &= \frac{\rho v_0^2 S_i}{2} \left(-\frac{\partial C_y}{\partial h} x_i + \frac{\partial C_{yi}}{\partial \alpha} \right); \\
\frac{\partial P_i}{\partial \varphi'} &= \frac{\rho v_0 S_i x_i}{2} \left(-\frac{\partial C_{yi}}{\partial \alpha} + \frac{\partial C_{yi}}{\partial \gamma} \right); \\
\frac{\partial P_i}{\partial h_W} &= \frac{\rho v_0^2 S_i}{2} \cdot \frac{\partial C_{yi}}{\partial h}; \\
\frac{\partial P_i}{\partial h'_W} &= -\frac{\rho v_0 S_i}{2} \cdot \frac{\partial C_{yi}}{\partial \gamma}; \\
\frac{\partial P_i}{\partial v_y} &= \frac{\rho v_0 S_i}{2} \cdot \frac{\partial C_{yi}}{\partial \alpha}; \\
\frac{\partial P_i}{\partial v_x} &= \frac{\rho S_i}{2} \left[-\frac{\partial C_{yi}}{\partial \alpha} (y' + x_i \varphi' - v_{yi}) + \frac{\partial C_{yi}}{\partial \gamma} (y' + x_i \varphi' - h'_{bi}) \right].
\end{aligned}\right\} \tag{3}$$

Assuming that

$$my'' = \sum^{i} \Delta P;$$

$$mr^2\varphi'' = \sum^{i} \Delta Px. \tag{4}$$

we get the system of equations /79

$$\begin{aligned} &y_0'' + a_1 y_0' + a_2 y_0 + a_3 \varphi' + a_4 \varphi = A; \\ &b_1 y_0' + b_2 y_0 + \varphi'' + b_3 \varphi' + b_4 \varphi = B, \end{aligned} \tag{5}$$

where

$$\left.\begin{aligned}
a_1 &= \frac{\varrho r^3}{2m} \sum^{i} \left(\frac{\partial C_y}{\partial \alpha} - \frac{\partial C_y}{\partial \gamma} \right) (1 + \bar{v}_x) S_0; \\
a_2 &= \frac{\varrho r^2}{2m} \sum^{i} \frac{\partial C_y}{\partial h} \frac{s_0}{b_0}; \\
a_3 &= \frac{\varrho r^3}{2m} \sum^{i} \left(\frac{\partial C_y}{\partial \alpha} - \frac{\partial C_y}{\partial \gamma} \right) (1 + \bar{v}_x) S_0 x_0; \\
a_4 &= \frac{\varrho r^3}{2m} \sum \left(\frac{\partial C_y}{\partial h} \cdot \frac{x_0}{b_0} - \frac{\partial C_y}{\partial \alpha} \right) S_0; \\
b_1 &= \frac{\varrho r^2}{2m} \sum^{i} \left(\frac{\partial C_y}{\partial \alpha} - \frac{\partial C_y}{\partial \gamma} \right) (1 + \bar{v}_x) S_0 x_0^2; \\
b_2 &= \frac{\varrho r^2}{2m} \sum \frac{\partial C_y}{\partial h} \cdot \frac{s_0 x_0}{b_0}; \\
b_3 &= \frac{\varrho r^3}{2m} \sum^{i} \left(\frac{\partial C_y}{\partial \alpha} - \frac{\partial C_y}{\partial \gamma} \right) (1 + \bar{v}_x) S_0 x_0^2; \\
b_4 &= \frac{\varrho r^3}{2m} \sum^{i} \left(\frac{\partial C_y}{\partial h} \cdot \frac{x_0}{b_0} - \frac{\partial C_y}{\partial \alpha} \right) S_0 x_0
\end{aligned}\right\} \tag{6}$$

(the subscripts are omitted here for the sake of simplicity).

The right hand members of the equations are determined by the formulas

$$\begin{aligned}
A &= \frac{\varrho r^3}{2m} \left[\sum^{i} \frac{\partial C_y}{\partial h} \cdot \frac{s_0}{b_0} \bar{h}_W - \sum^{i} \left(\frac{\partial C_y}{\partial \alpha} \bar{v}_y + \frac{\partial C_y}{\partial \gamma} \bar{h}'_W \right) (1 + \bar{v}_x) S_0 \right]; \\
B &= \frac{\varrho r^3}{2m} \left[\sum^{i} \frac{\partial C_y}{\partial h} \cdot \frac{s_0 x_0}{b_0} \bar{h}_W - \sum^{i} \left(\frac{\partial C_y}{\partial \alpha} \bar{v}_y + \frac{\partial C_y}{\partial \gamma} \bar{h}'_W \right) (1 + \bar{v}_x) S_0 x_0 \right].
\end{aligned} \tag{7}$$

The components of orbital velocity are

$$v_x = r_W \sigma_0 e^{\frac{2\pi}{\lambda_W} h_i} \cos 2\pi \left(\frac{x_i}{\lambda_W} - \frac{t}{\tau} \right);$$

$$v_y = r_W \sigma_0 e^{\frac{2\pi}{\lambda_W} h_i} \sin 2\pi \left(\frac{x_i}{\lambda_W} - \frac{t}{\tau} \right). \tag{8}$$

The elements of the wave relief are /80

$$\bar{h}_W = \frac{r_W}{r} \cos 2\pi \left(\frac{x}{\lambda_W} - \frac{t}{\tau} \right);$$
$$\bar{h}'_W = \frac{r_W}{r} \sin 2\pi \left(\frac{x}{\lambda_W} - \frac{t}{\tau} \right) v_0. \tag{9}$$

The following notation is used in these expressions: ρ is the mass density of the water; S_i is the area of the i-th hydrofoil; m is the mass of the vessel; r is the horizontal radius of gyration of the masses of the vessel; r_W is half the height of a wave; λ_W is the wave length;

$$y_0 = \frac{y}{r}; \qquad \bar{h}_i = \frac{h_i}{b_i}; \qquad x_{0i} = \frac{x_i}{r}$$

(b_i is the chord of the hydrofoil).

$$\sigma_0 = \sqrt{\frac{2\pi g}{\lambda_W}}; \qquad \bar{v}_x = \frac{v_x}{v_0}; \qquad \bar{v}_y = \frac{v_y}{v_0};$$
$$\tau = \frac{2\pi}{\sigma}; \qquad \sigma = \frac{2\pi}{\lambda_W}(v_0 \pm c),$$

where c is the velocity of motion of the waves, $c = 1.25\sqrt{\lambda b}$.

Since the coefficients a_i and b_i of system of equations (5) and their right hand members A and B are considerably dependent on displacements y and ϕ, the oscillations described by these equations are non-linear.

This system may be solved by numerical methods--the Newton method or the Runge-Kutta method. The second method, although more accurate, is somewhat cumbersome. Computational results have shown that the first method gives good results for a sufficiently large number of steps per period (24-32).

On the basis of the selected procedure for solving the system, two methods may be used for determining the derivatives $\frac{\partial C_y}{\partial h}, \frac{\partial C_y}{\partial \alpha}, \frac{\partial C_y}{\partial \gamma}$: 1) these derivatives should be calculated directly as a function of $\bar{h}$, λ, α, etc. on the basis of results obtained in hydrodynamics [2], and then introduced into the problem on oscillations in analytical or tabular form; 2) these same quantities may be approximately determined as ratios of the increments to the increments of the corresponding arguments $\left(\frac{\Delta C_y}{\Delta h}, \frac{\Delta C_y}{\Delta \alpha}, \frac{\Delta C_y}{\Delta \gamma}\right)$ for each step in the process of calculation. In the given case, the second method was used as the least cumbersome.

According to data in [2], the coefficient of lift which determines the vertical conditions is /81

$$C_y = \frac{a_\infty \psi}{1 + \frac{a_\infty \psi \zeta}{\pi \lambda}} (\alpha_{in} - \Delta\alpha_0). \tag{10}$$

Here $\alpha_\infty = 5.45$;

$$\psi = 1 + 4\gamma\tau + \tau^2 - 2\gamma\tau^3 + \frac{3}{4}\tau^4 - 2\gamma\tau^5 + 0(\tau^6); \tag{11}$$

$$\Delta\alpha_0 = -\frac{\varkappa\alpha_0}{\psi} + \frac{k\delta\tau^3}{2\psi}; \tag{12}$$

$$\zeta = 1 + F_1 - \frac{\gamma}{\lambda} F_2, \tag{13}$$

where

$$\tau = \sqrt{4\bar{h}^2 + 1} - 2\bar{h}; \tag{14}$$

$$\varkappa = \frac{1}{2}\tau^2 - \tau^4 + \frac{13}{16}\tau^6 - \frac{5}{8}\tau^8 + \frac{5}{16}\tau^{10}; \tag{15}$$

k is the characteristic of curvature of the intake surface of the hydrofoil, k = 1; δ is the relative thickness of the hydrofoil;

$$F_1 = 0{,}5\tau_\lambda^2 + 0{,}25\tau_\lambda^4 + 0{,}0625\tau_\lambda^6 + 0{,}0469\tau_\lambda^8 + 0{,}0257\tau_\lambda^{10} + 0{,}0288\tau_\lambda^{12}; \tag{16}$$

$$F_2 = (\tau_\lambda + \tau_\lambda^3 + 0{,}375\tau_\lambda^5 + 0{,}375\tau_\lambda^7 + 0{,}257\tau_\lambda^9 + 0{,}225\tau_\lambda^{11})\frac{\partial\tau_\lambda}{\partial\bar{h}}$$

In equations (16)

$$\tau_\lambda = \sqrt{4\bar{H}^2 + 1} - 2\bar{H}, \tag{17}$$

where

$$\bar{H} = \frac{h}{b\lambda}.$$

The value of C_{yi} is determined on each step with respect to the values of h_i, α_i and γ_i of the preceding step, and consequently $\Delta C_{yi} = C_{yi(n)} - C_{yi(n-1)}$ (n is the number of the step). The first derivative of displacement with respect to time may be approximately written in the form

$$y' \approx \frac{y_n - y_{n-1}}{\Delta t}, \tag{18}$$

while the second derivative may be written in the form

$$y'' \approx \frac{y_n - 2y_{n-1} + y_{n-2}}{(\Delta t)^2}. \tag{19}$$

Here y_n, y_{n-1} and y_{n-2} are displacements corresponding to the moments of time t_1, $t_1 - \Delta t$, $t_1 - 2\Delta t$.

On the basis of this representation, the equations of motion (after ele- /82
mentary transformation) may be rewritten as follows:

$$\begin{gathered}
y_{n+1}\left(1 + a_1\frac{\Delta t}{2}\right) + \varphi_{n+1}a_3\frac{\Delta t}{2} = A_n(\Delta t)^2 + y_n[2 - a_2(\Delta t)^2] - \\
- y_{n-1}\left(1 - a_1\frac{\Delta t}{2}\right) - \varphi_n a_4(\Delta t)^2 + \varphi_{n-1}a_3\frac{\Delta t}{2}; \\
y_{n+1}b_1\frac{\Delta t}{2} + \varphi_{n+1}\left(1 + a_1\frac{\Delta t}{2}\right) = B_n(\Delta t)^2 + \varphi_n[2 - b_4(\Delta t)^2] - \\
- \varphi_{n-1}\left(1 - b_3\frac{\Delta t}{2}\right) - y_n b_2(\Delta t)^2 + y_{n-1}b_1\frac{\Delta t}{2},
\end{gathered} \tag{20}$$

hence

$$
\left.
\begin{aligned}
y_{n+1} &= \frac{\begin{aligned}&A_n\left(1+b_3\frac{\Delta t}{2}\right)(\Delta t)^2 - B_n a_3\frac{(\Delta t)^3}{2} + \\ &+ y_n\left[(2-a_2(\Delta t)^2)\left(1+b_3\frac{\Delta t}{2}\right)+a_3 b_2\frac{(\Delta t)^3}{2}\right] - \\ &- y_{n-1}\left[\left(1-a_1\frac{\Delta t}{2}\right)\left(1+b_3\frac{\Delta t}{2}\right)+a_3 b_1\frac{(\Delta t)^2}{2}\right] - \\ &- \varphi_n\left[\left(1+b_3\frac{\Delta t}{2}\right)a_4(\Delta t)^2+(2-b_4(\Delta t)^2)\,a_3\frac{\Delta t}{2}\right]+\varphi_{n-1}a_3\Delta t\end{aligned}}{\left(1+a_1\frac{\Delta t}{2}\right)\left(1+b_3\frac{\Delta t}{2}\right)-a_3 b_1\frac{(\Delta t)^2}{4}}; \\
\varphi_{n+1} &= \frac{\begin{aligned}&B_n\left(1+a_1\frac{\Delta t}{2}\right)(\Delta t)^2 - A_n b_1\frac{(\Delta t)^3}{2} + \varphi_n\left[(2-b_4(\Delta t)^2)\times\right. \\ &\times\left.\left(1+a_1\frac{\Delta t}{2}\right)+a_4 b_1\frac{(\Delta t)^3}{2}\right] - \varphi_{n-1}\left[\left(1-b_3\frac{\Delta t}{2}\right)\times\right. \\ &\times\left.\left(1+a_1\frac{\Delta t}{2}\right)+a_3 b_1\frac{(\Delta t)^2}{4}\right] - y_n\left[\left(1-a_1\frac{\Delta t}{2}\right)b_2(\Delta t)^2+\right. \\ &+\left.(2-a_2(\Delta t)^2)\,b_1\frac{\Delta t}{2}\right]+y_{n-1}b_1\Delta t\end{aligned}}{\left(1+a_1\frac{\Delta t}{2}\right)\left(1+b_3\frac{\Delta t}{2}\right)-a_3 b_1\frac{(\Delta t)^2}{4}}.
\end{aligned}
\right\} \qquad (21)
$$

Thus the problem reduces to successive calculation of the quantities y_{n+1} and ϕ_{n+1} from their values in the preceding steps (y_n, y_{n-1}, ϕ_n, ϕ_{n-1} etc.). Consequently the initial conditions of the problem are indeterminate. We assume that the boat is moving on quiet water up to the time t = 0. At t = 0, the wave relief with ordinate h_B = 0 arrives at the point corresponding to the center of pressure of the forward hydrofoil (in the case of head-on waves) or the aft hydrofoil (in the case of overtaking waves). This moment corresponds to the zeroth step (n = 0). Displacements are equal to zero when n - 1 and n - 2. After determining y_0 and ϕ_0, we introduce the resultant values in formulas (21) in place of y_{n-1} and ϕ_{n-1}, compute the discrepencies on the first step, etc. In this case, free oscillations will be present together with forced oscillations during the first three or four full periods, although these free oscillations will be quickly attenuated due to the effect of intense damping, and the forced oscillations will remain to be determined. /83

Digital computer calculations have shown that non-linearity in the relationship between the coefficients of the equations and displacements has a considerable effect on the form of oscillations even in the case of small oscillation amplitudes in transresonance conditions. It has also been determined that fundamental oscillations with the period of the disturbing force are accompanied by oscillations with a period which is a multiple of the fundamental. In this case the amplitude of these oscillations is commensurate in certain cases with the amplitude of the fundamental oscillation.

This method may be used in calculations which account for lateral oscillations (motion on oblique courses, three-dimensional wave conditions, etc.).

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MOTION OF AN INCLINED WING CLOSE TO A DEFLECTOR

V. G. Belinskiy

ABSTRACT: The author studies the motion of a thin supporting surface inclined at an arbitrary angle to a solid deflector. A solution for a wing of eliptical planform is found on the basis of assumptions of lift line theory. Expressions are found for the coefficients of lift and banking moment.

The motion of an inclined hydrofoil beneath the free surface of a liquid /84
was studied by T. Nishiyama [3]. The motion of an inclined wing with optimum distribution of circulation was taken up by P. Zinchuk [2]. In this paper we shall give the results of an investigation of the effect of a deflector on the hydrodynamic characteristics of an inclined wing.

Let us consider the problem of steady-state motion with velocity v_0 and angle of attack α of a thin supporting surface inclined at an angle β to a flat solid deflector (Figure 1). The problem is solved by the acceleration potential method [1].

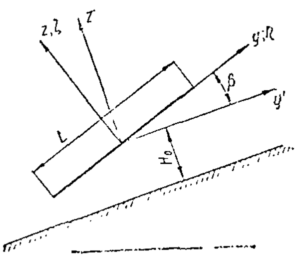

Figure 1.

For the velocity potential, we get the expression

$$\bar{\varphi} = -\frac{\lambda}{4\pi}\int_{-1}^{+1}\int_{-1}^{+1} \bar{\gamma}(\bar{0}) \left\{ \frac{(\bar{z}-\bar{\zeta})}{\lambda^2(\bar{y}-\bar{\eta})^2 + (\bar{z}-\bar{\zeta})^2}\left[\frac{(\bar{x}-\bar{\xi})}{\bar{r}} - 1\right] - \right.$$
$$\left. - \frac{(\lambda\bar{y} - \lambda\bar{\eta}q + \bar{\xi}p + \lambda\bar{n})\,p + (\bar{z} + \bar{\xi}q + \lambda\bar{\eta}p + \lambda\bar{m})\,q}{(\lambda\bar{y} - \lambda\bar{\eta}q + \bar{\xi}p + \lambda\bar{n})^2 + (\bar{z} + \bar{\xi}q + \lambda\bar{\eta}p + \lambda\bar{m})^2}\left[\frac{(\bar{x}-\bar{\xi})}{\bar{r}_1} - 1\right]\right\} d\bar{\xi}\, d\bar{\eta}. \quad (1)$$

Here

$$\bar{r} = \sqrt{(\bar{x}-\bar{\xi})^2 + \lambda^2(\bar{y}-\bar{\eta})^2 + (\bar{z}-\bar{\zeta})^2};$$

$$\bar{r}_1 = \sqrt{(\bar{x}-\bar{\xi})^2 + [\lambda\bar{y} - \lambda\bar{\eta}q + \bar{\xi}p + \lambda\bar{n}]^2 + [\bar{z} + \bar{\xi}q + \lambda\bar{\eta}p + \lambda\bar{m}]^2};$$

$$p = \sin 2\beta;$$

$$q = \cos 2\beta;$$

$$\bar{\gamma}(\bar{0}) + \frac{\gamma(0)}{v_0}; \quad \bar{0} = \frac{0}{v_0^2}; \quad \bar{\varphi} = \frac{2\varphi}{v_0 B}; \quad \lambda = \frac{L}{B};$$

$$\bar{x} = \frac{2x}{B}; \quad \bar{y} = \frac{2y}{\lambda B}; \quad \bar{z} = \frac{2z}{B};$$

$$\bar{m} = 2\bar{H}_0 \cos\beta;$$

$$\bar{n} = 2\bar{H}_0 \sin\beta,$$

where

$$\overline{H}_0 = \frac{2H_0}{\lambda B}.$$

Subjecting expression (1) to the boundary condition on the lifting surface

$$\overline{\varphi}_z = -\alpha \text{ when } \overline{z} = 0, \tag{2}$$

we get a two-dimensional singular integral equation for the problem

$$\begin{aligned}
\alpha = \frac{\lambda}{4\pi} \int_{-1}^{+1}\int_{-1}^{+1} \overline{\gamma}(\overline{0}) \cdot \Bigg\{ \frac{1}{\lambda^2 (\overline{y} - \overline{\eta})^2} \left[\frac{(\overline{x} - \overline{\xi})}{\sqrt{(\overline{x} - \overline{\xi})^2 + \lambda^2 (\overline{y} - \overline{\eta})^2}} - 1 \right] - \\
- \frac{(\overline{y} - \overline{\eta} q + \overline{n})^2 \lambda^2 q - (\overline{\eta} p + \overline{m})^2 \lambda^2 q - 2p\lambda^2 (\overline{y} - \overline{\eta} q + \overline{n})(\overline{\eta} p + \overline{m})}{[(\overline{y} - \overline{\eta} q + \overline{n})^2 \lambda^2 + (\overline{\eta} p + \overline{m})^2 \lambda^2]^2} \times \\
\times \left[\frac{(\overline{x} - \overline{\xi})}{\sqrt{(\overline{x} - \overline{\xi})^2 + (\overline{y} - \overline{\eta} q + \overline{n})^2 \lambda^2 + (\overline{\eta} p + \overline{m})^2 \lambda^2}} - 1 \right] + \\
+ \frac{(\overline{y} - \overline{\eta} q + \overline{n}) \lambda p + (\overline{\eta} p + \overline{m}) \lambda q}{(\overline{y} - \overline{\eta} q + \overline{n})^2 \lambda^2 + (\overline{\eta} p + \overline{m})^2 \lambda^2} \times \\
\times \left[\frac{(\overline{x} - \overline{\xi})(\overline{\eta} p + \overline{m}) \lambda}{[(\overline{x} - \overline{\xi})^2 + (\overline{y} - \overline{\eta} q + \overline{n})^2 \lambda^2 + (\overline{\eta} p + \overline{m})^2 \lambda^2]^{3/2}} \right] \Bigg\} d\overline{\xi} d\overline{\eta}.
\end{aligned} \tag{3}$$

At the present time, an equation of this type may be solved only by numerical methods. To obtain analytical results, we use the assumptions of Prandtl's lift line theory.

Accordingly, we shall assume that

$$\begin{aligned}
&(\overline{x} - \overline{\xi}) = 0; \\
&\int_{-1}^{+1} \overline{\gamma}(\overline{0})\, d\overline{\xi} = 2\lambda \overline{\Gamma}(\overline{\eta}),
\end{aligned} \tag{4}$$

where

$$\overline{\Gamma}(\overline{\eta}) = \frac{\Gamma(\eta)}{\lambda B v_0}.$$

Using expression (1) as a point of departure and taking relationships (4) into consideration, we may derive an expression for the velocity potential of an inclined wing of large aspect ratio moving close to a deflector on the basis of assumptions of lift line theory: /86

$$\bar{\varphi} = \bar{\varphi}_0 + \frac{\lambda^2}{2\pi}\int_{-1}^{+1}\bar{\Gamma}(\bar{\eta})\left\{\frac{(\bar{z}-\bar{\zeta})}{\lambda^2(\bar{y}-\bar{\eta})^2+(\bar{z}-\bar{\zeta})^2} - \right.$$
$$\left. - \frac{(\lambda\bar{y}-\lambda\bar{\eta}q+\bar{\zeta}p+\lambda\bar{n})p+(\bar{z}+\bar{\zeta}q+\lambda\bar{\eta}p+\lambda\bar{m})q}{(\lambda\bar{y}-\lambda\bar{\eta}q+\bar{\zeta}p+\lambda\bar{n})^2+(\bar{z}+\bar{\zeta}q+\lambda\bar{\eta}p+\lambda\bar{m})^2}\right\}d\bar{\eta}, \tag{5}$$

where $\bar{\phi}_0$ is the potential for the corresponding two-dimensional problem.

Subjecting this expression to boundary condition (2) and relating the value of $\bar{\phi}_0$ to the value of the circulation on the wing, we get an equation for determining the circulation on an inclined wing close to a deflector:

$$\bar{\Gamma}(\bar{y}) = \frac{a_0\psi(\bar{y})}{2\lambda(\bar{y})}\left\{\alpha + \frac{1}{2\pi}\int_{-1}^{+1}\bar{\Gamma}(\bar{\eta})\left[\frac{1}{(\bar{y}-\bar{\eta})^2} - \right.\right.$$
$$\left.\left. - \frac{[q\bar{\eta}^2-2\bar{y}\bar{\eta}+q\bar{y}^2-2n(\bar{y}+\bar{\eta})-4\bar{H}_0^2]}{[\bar{\eta}^2-2q\bar{y}\bar{\eta}+\bar{y}^2+2n(\bar{y}+\bar{\eta})+4\bar{H}_0^2]^2}\right]d\bar{\eta}\right\}. \tag{6}$$

Here a_0 is the tangent of the slope of the curve $C_y = C_y(\alpha)$ for an unbounded fluid, where the theoretical value of this slope determined from the solution of the corresponding two-dimensional problem is equal to 2π; $\psi(\bar{y})$ is a function which accounts for the change in a_0 in a bounded flow.

When $\beta = 0°$, equation (6) reduced to an equation for a horizontal wing close to a deflector:

$$\bar{\Gamma}(\bar{y})_{\beta=0} = \frac{a_0\psi(\bar{y})}{2\lambda(\bar{y})}\left\{\alpha + \frac{1}{2\pi}\int_{-1}^{+1}\bar{\Gamma}(\bar{\eta})\left[\frac{1}{(\bar{y}-\bar{\eta})^2} - \frac{(\bar{y}-\bar{\eta})^2-4\bar{H}_0^2}{[(\bar{y}-\bar{\eta})^2+4\bar{H}_0^2]^2}\right]d\bar{\eta}\right\},$$

and when $\beta = 90°$, this equation reduces to an equation for a vertical wing close to a deflector

$$\bar{\Gamma}(\bar{y})_{\beta=90°} = \frac{a_0\psi(\bar{y})}{2\lambda(\bar{y})}\left\{\alpha + \frac{1}{2\pi}\int_{-1}^{+1}\bar{\Gamma}(\bar{\eta})\left[\frac{1}{(\bar{y}-\bar{\eta})^2} + \frac{1}{(\bar{y}+\bar{\eta}+2\bar{H}_0)^2}\right]d\bar{\eta}\right\}.$$

Let use the notation $G(\bar{y}, \bar{\eta})$ for the kernel of equation (6). Let us consider the integral operator

$$L(\bar{y}) = \frac{1}{2\pi}\int_{-1}^{+1} \bar{\Gamma}(\bar{\eta})\, G(\bar{y}, \bar{\eta})\, d\bar{\eta}, \tag{7}$$

which may be represented as the sum of a symmetric term and an asymmetric term:

$$L(\bar{y}) = L_s(\bar{y}) + L_a(\bar{y}) = \frac{L(\bar{y}) + L(-\bar{y})}{2} + \frac{L(\bar{y}) - L(-\bar{y})}{2}. \tag{8}$$

We may write /87

$$L_s(\bar{y}) = \frac{1}{2\pi}\int_{-1}^{+1} \bar{\Gamma}(\bar{\eta})\, G_s(\bar{y}, \bar{\eta})\, d\bar{\eta};$$

$$L_a(\bar{y}) = \frac{1}{2\pi}\int_{-1}^{+1} \bar{\Gamma}(\bar{\eta})\, G_a(\bar{y}, \bar{\eta})\, d\bar{\eta}),$$

where

$$G_s(\bar{y}, \bar{\eta}) = \frac{G(\bar{y}, \bar{\eta}) + G(-\bar{y}, \bar{\eta})}{2}; \quad G_a(\bar{y}, \bar{\eta}) = \frac{G(\bar{y}, \bar{\eta}) - G(-\bar{y}, \bar{\eta})}{2}.$$

The kernels $G_s(\bar{y}, \bar{\eta})$ and $G_a(\bar{y}, \bar{\eta})$ may be treated in turn as a sum of symmetric and asymmetric terms:

$$\begin{aligned} G_s(\bar{y}, \bar{\eta}) &= G_{ss}(\bar{y}, \bar{\eta}) + G_{sa}(\bar{y}, \bar{\eta}); \\ G_a(\bar{y}, \bar{\eta}) &= G_{as}(\bar{y}, \bar{\eta}) + G_{aa}(\bar{y}, \bar{\eta}), \end{aligned} \tag{9}$$

where

$$\begin{aligned} G_{ss}(\bar{y}, \bar{\eta}) &= \frac{1}{4}[G(\bar{y}, \bar{\eta}) + G(-\bar{y}, \bar{\eta})] + \frac{1}{4}[G(\bar{y}, -\bar{\eta}) + G(-\bar{y}, -\bar{\eta})]; \\ G_{sa}(\bar{y}, \bar{\eta}) &= \frac{1}{4}[G(\bar{y}, \bar{\eta}) + G(-\bar{y}, \bar{\eta})] - \frac{1}{4}[G(\bar{y}, -\bar{\eta}) + G(-\bar{y}, -\bar{\eta})]; \\ G_{as}(\bar{y}, \bar{\eta}) &= \frac{1}{4}[G(\bar{y}, \bar{\eta}) - G(-\bar{y}, \bar{\eta})] + \frac{1}{4}[G(\bar{y}, -\bar{\eta}) - G(-\bar{y}, -\bar{\eta})]; \\ G_{aa}(\bar{y}, \bar{\eta}) &= \frac{1}{4}[G(\bar{y}, \bar{\eta}) - G(-\bar{y}, \bar{\eta})] - \frac{1}{4}[G(\bar{y}, -\bar{\eta}) - G(-\bar{y}, -\bar{\eta})]. \end{aligned} \tag{10}$$

Separating the circulation into symmetric and asymmetric terms

$$\bar{\Gamma}(\bar{\eta}) = \bar{\Gamma}_s(\bar{\eta}) + \bar{\Gamma}_a(\bar{\eta}), \tag{11}$$

we get

$$L_c(\bar{y}) = \frac{1}{2\pi}\int_{-1}^{+1}[\bar{\Gamma}_s(\bar{\eta})\cdot G_{ss}(\bar{y},\bar{\eta}) + \bar{\Gamma}_a(\bar{\eta})\cdot G_{ca}(\bar{y},\bar{\eta})]\,d\bar{\eta};$$
$$L_a(\bar{y}) = \frac{1}{2\pi}\int_{-1}^{+1}[\bar{\Gamma}_s(\bar{\eta})\cdot G_{as}(\bar{y},\bar{\eta}) + \bar{\Gamma}_a(\bar{\eta})\cdot G_{aa}(\bar{y},\bar{\eta})]\,d\bar{\eta}. \tag{12}$$

The function $\psi(y)$ is treated analogously: /88

$$\psi(\bar{y}) = \psi_s(\bar{y}) + \psi_a(\bar{y}). \tag{13}$$

Thus we arrive at the system of equations

$$\left.\begin{aligned}
\bar{\Gamma}_s(\bar{y}) = \frac{a_0}{2\lambda(\bar{y})}\psi_c(\bar{y})\Bigg\{\alpha + \frac{1}{2\pi}\int_{-1}^{+1}[\bar{\Gamma}_s(\bar{\eta})\cdot G_{ss}(\bar{y},\bar{\eta}) + {} \\
+ \bar{\Gamma}_a(\bar{\eta})\,G_{sa}(\bar{y},\bar{\eta})]\,d\bar{\eta} + {} \\
+ \frac{\psi_a(\bar{y})}{\psi_s(\bar{y})}\cdot\frac{1}{2\pi}\int_{-1}^{+1}[\bar{\Gamma}_s(\bar{\eta})\,G_{as}(\bar{y},\bar{\eta}) + {} \\
+ \bar{\Gamma}_a(\bar{\eta})\,G_{aa}(\bar{y},\bar{\eta})]\,d\bar{\eta}\Bigg\}; \\
\bar{\Gamma}_a(\bar{y}) = \frac{a_0}{2\lambda(\bar{y})}\psi_a(\bar{y})\Bigg\{\alpha + \frac{1}{2\pi}\int_{-1}^{+1}[\bar{\Gamma}_s(\bar{\eta})\,G_{ss}(\bar{y},\bar{\eta}) + {} \\
+ \bar{\Gamma}_a(\bar{\eta})\,G_{sa}(\bar{y},\bar{\eta})]\,d\bar{\eta} + {} \\
+ \frac{\psi_s(\bar{y})}{\psi_a(\bar{y})}\cdot\frac{1}{2\pi}\int_{-1}^{+1}[\bar{\Gamma}_s(\bar{\eta})\,G_{as}(\bar{y},\bar{\eta}) + {} \\
+ \bar{\Gamma}_a(\bar{\eta})\,G_{aa}(\bar{y},\bar{\eta})]\,d\bar{\eta}\Bigg\}
\end{aligned}\right\} \tag{14}$$

Let us give one approximate solution for this system. For this purpose we take the ratios of the functions $\frac{\psi_c(\bar{y})}{\lambda(\bar{y})}$ and $\frac{\psi_a(\bar{y})}{\lambda(\bar{y})}$ in the form

$$\frac{\psi_s(\bar{y})}{\lambda(\bar{y})} = a\sqrt{1-\bar{y}^2}; \quad \frac{\psi_a(\bar{y})}{\lambda(\bar{y})} = b\bar{y}\sqrt{1-\bar{y}^2}, \tag{15}$$

where

$$a = \frac{4}{\pi\lambda_0}\,\psi_c(\beta), \quad b = \frac{4}{\pi\lambda_0}\,\psi_a(\beta).$$

We shall seek the solution of system (14) in the form

$$\left.\begin{aligned} \bar{\Gamma}_s(\bar{y}) &= A\sqrt{1-\bar{y}^2};\\ \Gamma_a(\bar{y}) &= B\bar{y}\sqrt{1-\bar{y}^2}. \end{aligned}\right. \tag{16}$$

The solution of system (14) with regard to expressions (15) and (16) gives the following values for the constants A and B: /89

$$\begin{gathered} A = \frac{\frac{a_0 a}{2}\alpha\left[1+\frac{a_0 a}{2}\zeta_4-\frac{a_0 b}{4}\zeta_6\right]+\frac{a_0 b}{2}\alpha\left[\frac{a_0 a}{4}\zeta_2-\frac{a_0 b}{8}\zeta_1\right]}{\left[1+\frac{a_0 a}{2}\zeta_1-\frac{a_0 b}{4}\zeta_6\right]\left[1+\frac{a_0 a}{4}\zeta_1-\frac{a_0 b}{4}\zeta_3\right]-\left[\frac{a_0 a}{4}\zeta_2-\frac{a_0 b}{8}\zeta_1\right]\left[a_0 a\zeta_3-\frac{a_0 b}{4}\zeta_5\right]};\\ B = \frac{\frac{a_0 a}{2}\alpha\left[a_0 a\zeta_3-\frac{a_0 b}{4}\zeta_5\right]+\frac{a_0 b}{2}\alpha\left[1+\frac{a_0 a}{4}\zeta_1-\frac{a_0 b}{4}\zeta_3\right]}{\left[1+\frac{a_0 a}{2}\zeta_1-\frac{a_0 b}{4}\zeta_6\right]\left[1+\frac{a_0 a}{4}\zeta_1-\frac{a_0 b}{4}\zeta_3\right]-\left[\frac{a_0 a}{4}\zeta_2-\frac{a_0 b}{8}\zeta_1\right]\left[a_0 a\zeta_3-\frac{a_0 b}{4}\zeta_5\right]}, \end{gathered} \tag{17}$$

where

$$\left.\begin{aligned} \zeta_1 &= 1-\frac{1}{2}\tau^2-\left[\frac{1}{4}+\frac{3}{2}\sin^2\beta\right]\tau^4-\left[\frac{1}{16}+\frac{17}{8}\sin^2\beta-\frac{9}{4}\sin^4\beta\right]\tau^6-\left[\frac{3}{64}-\frac{303}{32}\sin^2\beta+6\sin^4\beta+3\sin^6\beta\right]\tau^8;\\ \zeta_2 &= \zeta_3 = -\frac{1}{4}\sin\beta\tau^3+\left[-\frac{3}{8}\sin\beta-\sin^3\beta\right]\tau^5+\left[-\frac{21}{64}\sin\beta+\frac{89}{16}\sin^3\beta-\frac{3}{2}\sin^5\beta\right]\tau^7+\left[\frac{61}{64}\sin\beta-\frac{93}{16}\sin^3\beta+\frac{3}{2}\sin^5\beta\right]\tau^9;\\ \zeta_4 &= 1-\frac{3}{8}\tau^4-\left[\frac{1}{4}+\frac{5}{2}\sin^2\beta-4\sin^4\beta\right]\tau^6-\left[\frac{5}{64}-\frac{27}{4}\sin^2\beta+\frac{87}{8}\sin^4\beta-4\sin^6\beta\right]\tau^8;\\ \zeta_5 &= 1-\frac{1}{2}\tau^2-\left[\frac{1}{8}+\frac{9}{4}\sin^2\beta\right]\tau^4-\left[-\frac{1}{32}+\frac{41}{16}\sin^2\beta-\frac{21}{4}\sin^4\beta\right]\tau^6-\left[\frac{9}{64}-\frac{481}{32}\sin^2\beta+\frac{87}{8}\sin^4\beta+\frac{27}{4}\sin^6\beta\right]\tau^8;\\ \zeta_6 &= -\frac{1}{4}\sin\beta\cdot\tau^3-\left[\frac{3}{4}\sin\beta+\sin^3\beta\right]\tau^5+\left[-\frac{9}{8}\sin\beta+\frac{79}{8}\sin^3\beta-\frac{21}{8}\sin^5\beta\right]\tau^7+\left[\frac{17}{8}\sin\beta-\frac{87}{8}\sin^3\beta+\frac{21}{8}\sin^5\beta\right]\tau^9. \end{aligned}\right\} \tag{18}$$

In these formulas $\tau = \sqrt{\lambda_0^2 \bar{H}_0^2 + 1} - \lambda_0 \bar{H}_0$. /90

The coefficients of lift and banking moment are defined by the expressions

$$C_{z_{\bar{H}}} = \lambda_0 \int_{-1}^{+1} \bar{\Gamma}_S(\bar{y})\, d\bar{y}; \tag{19}$$

$$C_{mx_{\bar{H}}} = \frac{1}{2} \lambda_0 \int_{-1}^{+1} \bar{y} \cdot \bar{\Gamma}_S(\bar{y})\, d\bar{y}.$$

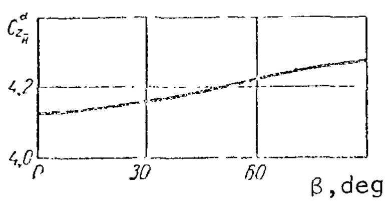

Figure 2.

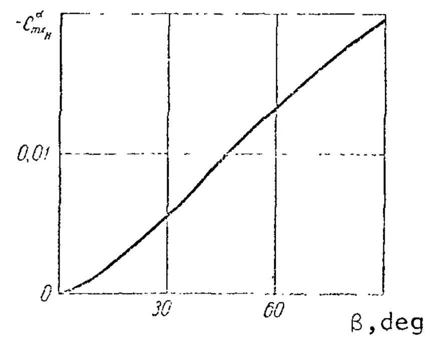

Figure 3.

Taking equations (16) into consideration, we get

$$C_{z_{\bar{H}}} = \frac{1}{2} \pi \lambda_0 A,$$

$$C_{m_{x_{\bar{H}}}} = \frac{1}{16} \pi \lambda_0 B,$$

where A and B are defined by expressions (17).

Given in Figure 2 is a graph of the function $C^{\alpha}_{z_{\bar{h}}}$ as related to the angle of inclination β for a wing with aspect ratio $\lambda_0 = 5$, and relative distance from the deflector $\bar{H}_0 = 1.1$.

An analogous graph for the function $C^{\alpha}_{mx_{\bar{h}}}$ is given in Figure 3.

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CALCULATION OF THE CHARACTERISTICS OF A LIFTING SYSTEM WITH ELASTIC ELEMENTS

B. S. Berkovskiy

ABSTRACT: Data are given from influence function calculations which characterize the interaction of elastic and inelastic elements of a lifting system of infinite span with each other and with a deflector of free surface.

We shall define a lifting system as a set of closed or open lifting elements. In this case a wing with leading and trailing retractible flaps or a wing and tail assembly are systems of lifting surfaces. The interaction of the rigid elements of the lifting system with each other and with the boundaries of the flow causes considerable changes in the force and moment characteristics of isolated surfaces in an unbounded fluid. Elasticity may also lead in its turn to considerable changes in the aerohydrodynamic coefficients, a fact which may be used for controlling the behavior of the characteristics in which we are interested. From this standpoint, investigation of the motion of a system of contours with elastic elements and the motion of completely elastic systems is of both theoretical and practical interest. /91

In this paper, solutions found previously[1] are taken into consideration and data are given from the calculation of influence functions which characterize the interaction of elastic and inelastic elements of a lifting system of infinite span with each other and with a deflector or free surface.

The results may be used in the calculation of bodies in an infinite or bounded fluid: A wing with passive mechanization with regard to structural elasticity, an aircraft lift system, the "ducts" system, the tandem polyplane, and also in the design of special autostable lifting systems.

The change in properties of an individual element or of the lifting system as a whole may be characterized by influence functions--interactions of the form /92

$$\frac{P}{P_0} = \psi, \tag{1}$$

where P is the state to be considered; P_0 is the known state of the element or system.

[1]B. S. Berkovskiy, reports at the 17th Scientific and Technical Conference on Ship Theory (Krylov Lectures), Leningrad, 1967.

In the general case, the state is characterized by a number of parameters:

$$P = P(p_i). \tag{2}$$

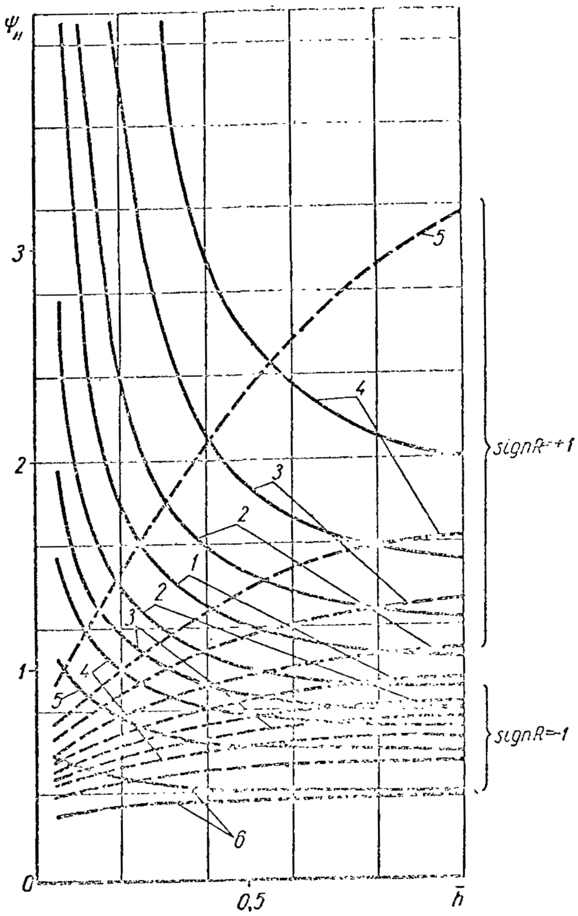


Figure 1. The Influence Function of the Boundary for an Isolated Elastic Contour Where sign Fr = -1 (Solid Lines) and sign Fr = +1 (Broken Lines): 1, H = 0; 2, H = 0.1; 3, H = 0.2; 4, H = 0.3; 5, H = 0.5; 6, H = 1.

In our specific case, the force characteristic P is determined by the parameters of distance from the boundary h, rigidity EI, relative position and the ratio of the dimensions of the elements d which take on both intermediate and limiting values: /94

$$P = P(h, EI, d), \tag{3}$$

i.e. the influence function of the deflector is

$$\psi = \psi(h, EI, d).$$

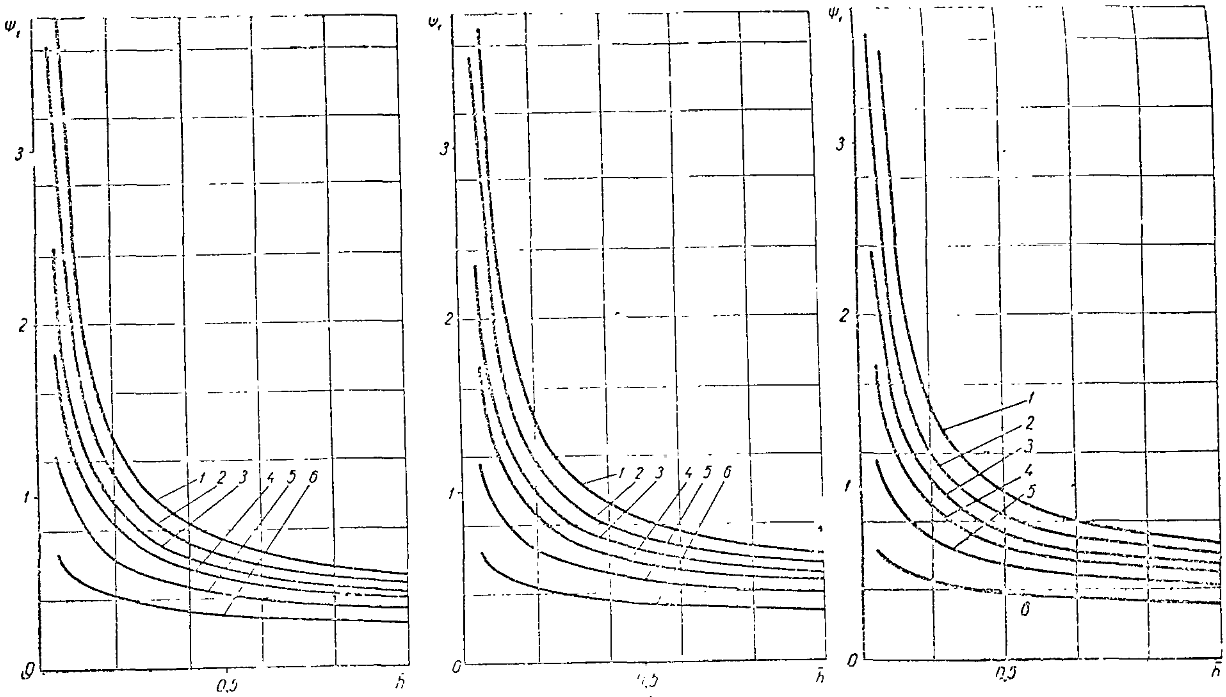


Figure 2. Influence Function of a Deflector for Element No. 1 Where sign Fr = -1: a, t = 1; b, t = 2; c, t = 3 (The Designations For H are the Same as in Figure 1).

The results of investigations for an isolated element are shown in Figure 1. In the case of a rigid plate in an unbounded fluid, $\psi = 1$, and in all other cases $0 < \psi \gtrless 1$; the coefficient of lift is $C_y = a_\infty \psi \alpha_{ef}$.

The influence function of the system is determined from the influence function for the element:

$$\psi_\Sigma^{0\Sigma} = \frac{P_\Sigma}{P_{0\Sigma}} = \frac{\psi_n + \sum_1^{m}{}' t_{in} k_{in} \psi_i}{1 + \sum_1^{m}{}' t_{in} k_{in}}, \tag{4}$$

where

$$t_{in} = \frac{\alpha_i}{\alpha_n}; \quad k_{in} = \frac{k_i}{k_n};$$

$$\Sigma' = \Sigma \text{ when } i \neq n$$

(n is the index of the element, k_i is the chord of the i-th element).

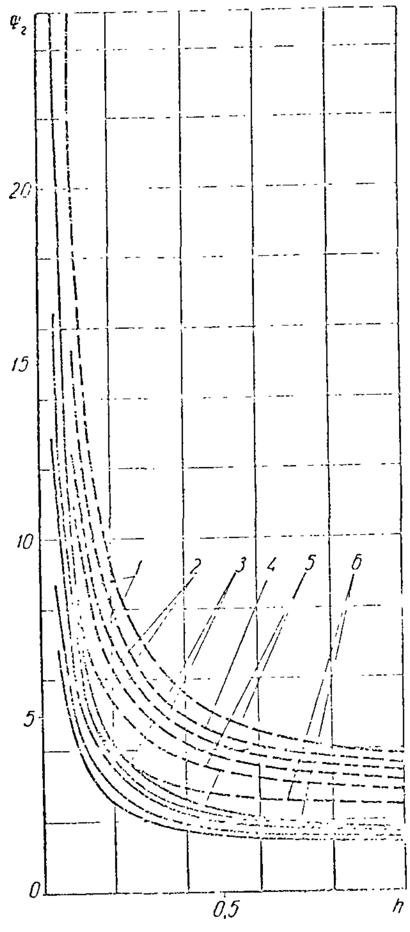


Figure 3. Influence Function of A Deflector for Element No. 2 Where sign Fr = -1 and t = 1 (Solid Lines), t = 3 (Broken Lines). The Designations for H are the Same as in Figure 1.

For closed equally large plates, the influence functions of the elements are given in Figures 2-3, and the influence functions of the system are given in Figure 4. The following notation is used in these figures:

$$H = -\frac{k^3 \varrho v_\infty^2}{8EI}; \quad H_1 = \frac{k^4 \gamma_B}{8EI};$$

$$\tilde{H} = \frac{H}{6}; \quad \operatorname{sign} F_2 = +1$$

beneath a free surface; sign Fr = -1 above a deflector; sign R = +1 and sign R = -1 correspond to displacements of the flap on the trailing and leading edges; α_i are the angles of attack of the edge. /96

Analysis of the results leads to the following conclusions.

For an isolated elastic element in an unbounded fluid in the case of a flap in the leading edge (sign R = -1) there is a drop in lift as compared with a rigid plate due to a reduction in the local angles of attack in the direction of the trailing edge. Consequently, we may expect a shift in the center of pressure toward the leading edge of the wing and a corresponding change in the coefficient of longitudinal static stability $\partial m_z/\partial C_y$. The reverse process takes place in the case of a flap in the trailing edge (sign R = +1).

The lift of an elastic contour increases less than for a rigid contour close to a deflector when sign R = -1. As the distance to the deflector decreases, relative losses increase due to a reduction in the ratio $\partial\psi/\partial\bar{h}$ --the quantity which characterizes the margin of static stability of the wing with respect to height. When sign R = 1, the lift of an elastic contour and the quantity $\partial\psi/\partial\bar{h}$ increase with a reduction in the distance to the deflector in comparison with a rigid contour.

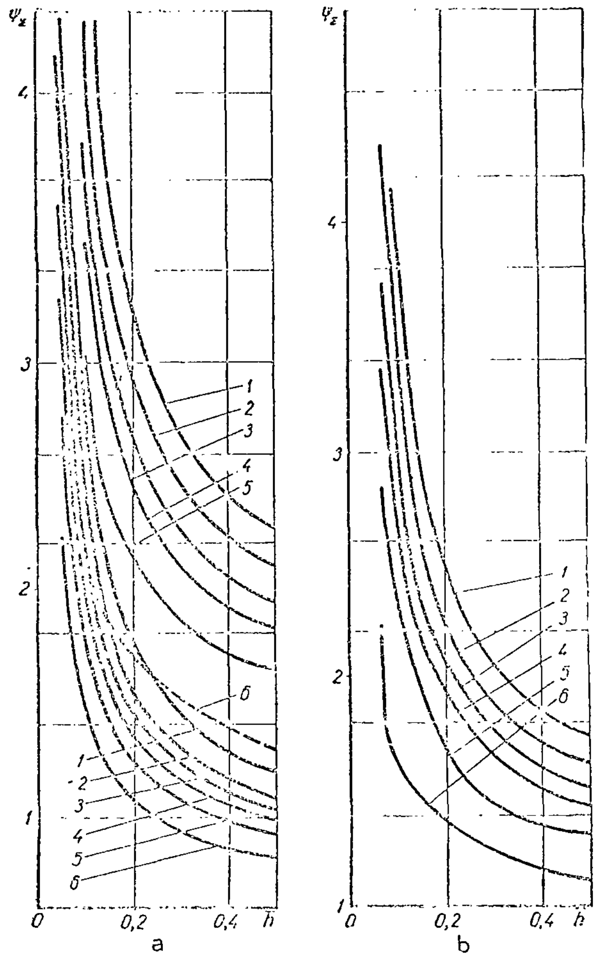


Figure 4. Influence Function of a Deflector or a Lifting System Where sign Fr = -1; sign R_1 = -1: a, t = 1 (Solid Lines), t = 3 (Broken Lines); b, t = 2 (Designations for H are the Same as in Figure 1).

Beneath a free surface in the case where sign R = -1, there is a reduction in the lift and the quantity $\partial\psi/\partial\bar{h}$ as the surface is approached. When sign R = +1, there is a transition in the results to the case of an infinite fluid as the load is increased.

General conclusions on closed and open supporting systems may be drawn on the basis of those enumerated above. Some additional conclusions are of interest.

A deflector has a weaker effect on a rigid horizontal element of control or mechanization with lower power percentage wise. For instance 20% of the control surfaces which produce 5% of the load in an unbounded fluid show practically no increase in load close to a deflector for the heights at which the system is operated. There is a sharp increase in lift on the remaining control surfaces, i.e. the relative loading of a 20% control surface (aileron, elevator) drops appreciably close to a deflector (by a factor of 1.7 when h_{syst} = 0.2). Consequently the percentage ratio of the areas of ailerons, flaps, elevators and wing for ground effect machines will be insufficient for providing the same control efficiency as compared with aircraft in an infinite fluid. Accounting for structural elasticity requires additional determination of the ratio between the areas of elements.

EFFECT OF PLANFORM ON THE AERODYNAMIC CHARACTERISTICS OF A WING CLOSE TO A DEFLECTOR

A. N. Lukashenko, Yu. I. Laptev
and A. G. Novikov

ABSTRACT: The authors describe the results of experimental investigations of wing-plates with various planforms in a wing tunnel. Relationships are given for the coefficients $C_y = f(\alpha, \bar{h})$, $C_x = f(\alpha, \bar{h})$ and $k = f(\lambda, \bar{h})$, as well as an equation which summarizes the results of the theoretical calculations.

The experimental data given below were recorded in a T-1 wind tunnel at a wind speed of $v = 30$ m/sec, ($Re \approx 4.6 \cdot 10^5$). The working section of the wind tunnel was open. The flow turbulence was $\varepsilon = 1.2\%$. /97

Investigations have shown that a solid fixed deflector has little effect on flow obliquity, and therefore it was completely permissible to introduce conventional corrections for flow obliquity and the effect of jet boundaries.

A more complex problem is the violation of boundary conditions on the surface of the fixed deflector which apparently results in somewhat of an elevation in the measured drag forces Q, particularly when the distance $\bar{h}$ between the wing and the deflector is small. The distances from the deflector were measured from the trailing edge of the wing in the root section.

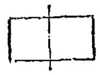
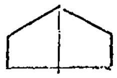
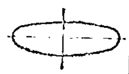
The wings were made in the form of metal plates of thickness $\bar{c} = 2\%$. The geometric characteristics of these wings are shown in the table.

The aspect ratio of the wings λ_0 was varied from 1.25 to 3 since these quantities were of the greatest practical interest.

According to curves for $C_y = f(\alpha)$ (Figure 1) there is an increase in the lift coefficient as the wing approaches the deflector for all angles of attack[1].

The best wings with respect to lift-drag ratio k (Figure 2) were those of trapezoidal planform. Maximum lift-drag ratio is reached for all wings at an angle of attack of approximately 4°. As $\bar{h}$ is reduced, there is somewhat of a reduction in the angle of maximum lift-drag ratio, and this angle reaches 3° for some wings when $\bar{h} = 0.1$. /100

[1]Curves are not given in this paper for $C_x = f(\alpha)$ and $Cm_z = f(\alpha)$.

Model No.	Planform	Span, mm	Root Chord b_r, mm	Tip Chord b_t, mm	Planform Area S, m^2	Aspect Ratio λ	Taper $\eta = b_r/b_t$	Angle χ, deg	a мм	b. мм
1		450	350	350	0,1575	1,25	1	—	—	—
2		460	230	230	0,1058	2	1	—	—	—
3		690	230	230	0,1585	3	1	—	—	—
4		460	275	185	0,1058	2	1,485	21°50′	—	—
5		690	300	160	0,1685	3	1,875	21°50′	—	—
6		690	230	0	0,1585	3	—	53°10′	—	—
7		460	293	0	0,1058	2	—	—	230	146,5

Calculation by formulas derived by A. N. Panchenkov and A. I. Yukhimenko in [2] shows satisfactory agreement between theoretical and experimental data.

Theoretical curves for $C_y = f(\alpha)$ are shown by the light lines in Figure 1b. The calculations were based on the formulas

$$C_y = \frac{\psi a_\infty}{C_2(\lambda_0)\left[1 + \frac{\psi a_\infty}{C_2(\lambda_0)\,\pi\lambda_0}\zeta\right]}\alpha;$$

$$C_{xi} = \frac{C_y^2}{\pi\lambda_0}\zeta_0;$$

$$\zeta = [1 + C_1(\lambda_0)]\zeta_0;$$

$$\zeta_0 = 1 - \frac{1}{2}\tau_\lambda^2 - 0{,}25\tau_\lambda^4 - 0{,}0625\tau_\lambda^6 - 0{,}046\tau_\lambda^8 - 0{,}0237\tau_\lambda^{10} - 0{,}0188\tau_\lambda^{12};$$

$$\psi = 1 + \tau^2 + 0(\tau^4);$$

$$\tau = \sqrt{4\bar{h}^2 + 1} - 2\bar{h}; \quad \tau_\lambda = \sqrt{4\bar{H}_0^2 + 1} - 2\bar{H}_0.$$

Here $\overline{h} = h/b_{av}$; $\overline{H}_0 = h/2l$, where b_{av} is the average chord of the wing; l is half the span of the wing. /101

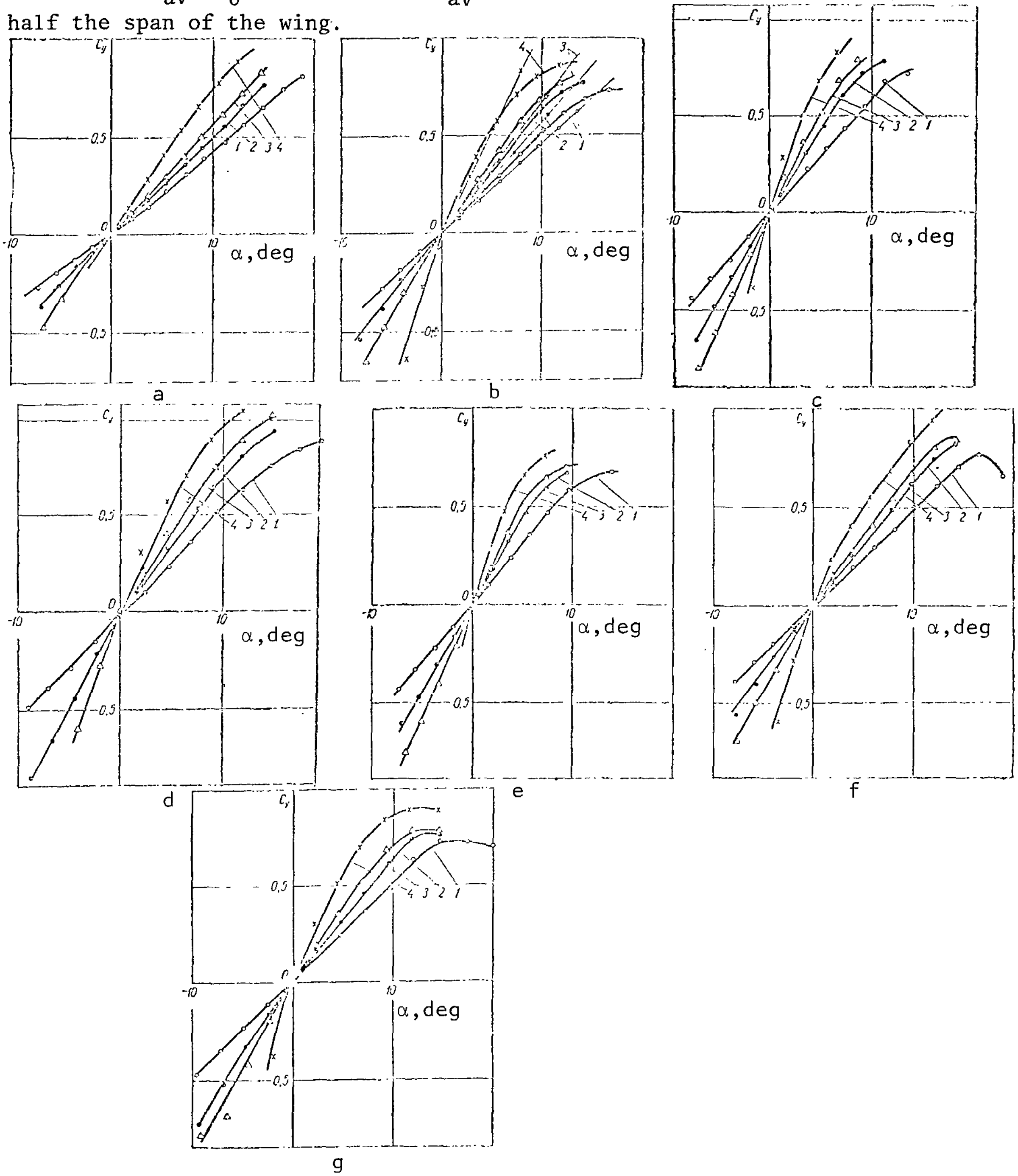


Figure 1. Curves for $C_y = f(\alpha)$: a, Model No. 1; b, Model No. 2; c, Model No. 3; d, Model No. 4; e, Model No. 5; f, Model No. 6; g, Model No. 7 (1, $\overline{h} = 0$; 2, $\overline{h} = 0.5$; 3, $\overline{h} = 0.3$; 4, $\overline{h} = 0.1$).

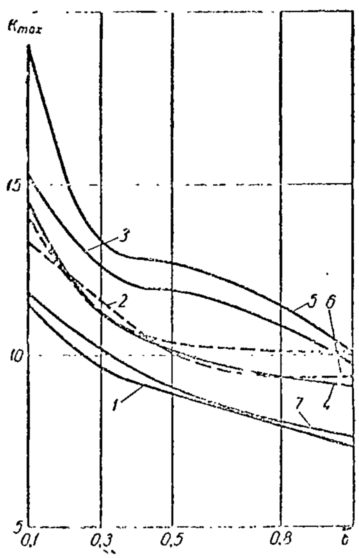


Figure 2. Curves for $k_{max} = f(\overline{h})$. Curves 1-7 Correspond to the Numbers of the Models in the Table.

The functions

$$C_1(\lambda_0) = e^{-b\lambda_0} \quad (b = 0{,}2599);$$

$$C_2(\lambda_0) = e^{-\frac{1}{a\lambda_0}} \quad (a_0 = 3{,}847)$$

with regard to arbitrariness of the aspect ratio.

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FLUTTERING WING CLOSE TO THE SURFACE OF A LIQUID OF FINITE DEPTH

A. N. Lukashenko

ABSTRACT: An expression is derived for the influence function of a deflector $\psi = C_y/C_{y\infty}$ for a fluttering wing close to the surface of a liquid of finite depth. The problem is solved by the method developed by the author in his paper on unsteady motion of a wing close to a deflector with low Strouhal numbers, and also by the method of analogy with the problem of a fluid of infinite depth.

An expression was derived in [3] for the singular integral operator $L\gamma$ for a fluttering wing close to a deflector with small Strouhal numbers k: /102

$$L\gamma = \frac{1}{2\pi}\int_{-1}^{+1} \gamma_1(s)\left[\frac{1}{x-s} + G(x-s)\right] ds. \tag{1}$$

The problem of the motion of a wing close to the surface of a liquid of finite depth may be solved as follows, using the method of solution proposed in [3].

The regular component G(x - s) of the kernel of equation (1) in this case will take the following form [2]:

when Fr → ∞

$$G_{\bar{h}\bar{h}_0}(x-s) = -\frac{x-s}{(x-s)^2+16(\bar{h}_0-\bar{h})^2} + 2\int_0^\infty \frac{e^{-2\lambda\bar{h}_0}\,\mathrm{sh}^2\,2\lambda(\bar{h}_0-\bar{h})}{\mathrm{ch}\,2\lambda\bar{h}_0}\sin\lambda(x-s)\,d\lambda;$$

when Fr → 0

$$G_{\bar{h}\bar{h}_0}(x-s) = -\frac{x-s}{(x-s)^2+16(\bar{h}_0-\bar{h})^2} - 2\int_0^\infty \frac{e^{-2\lambda\bar{h}_0}\,\mathrm{sh}^2\,2\lambda(\bar{h}_0-\bar{h})}{\mathrm{sh}\,2\lambda\bar{h}_0}\sin\lambda(x-s)\,d\lambda.$$

Here $\bar{h}_0$ is the relative depth of the liquid, $\bar{h}_0 = h_0/b$; $\bar{h}$ is the relative distance of the wing from the deflector, $\bar{h} = h/b$; b is the chord of the wing.

Without disturbing the generality of the problem let us consider the case Fr → 0 (Fr → ∞ may be derived analogously without particular difficulties). /103

The operator Lγ may be presented in the form of two components:

$$L\gamma = L_0\gamma + L_1\gamma, \tag{2}$$

where $L_0\gamma$ is the singular characteristic operator, $L_0\gamma = \frac{1}{2\pi}\int\limits_{-1}^{+1}\frac{\gamma_1(s)}{x-s}ds$;

Lγ is the regular integral operator,

$$L_1\gamma = \frac{1}{2\pi}\int\limits_{-1}^{+1}\gamma_1(s)\left[-\frac{x-s}{(x-s)^2+16(\bar{h}_0-\bar{h})^2}-\right.$$
$$\left.-2\int\limits_0^\infty\frac{e^{-2\lambda\bar{h}_0}\,\mathrm{sh}^2\,2\lambda(\bar{h}_0-\bar{h})}{\mathrm{sh}\,2\lambda\bar{h}_0}\sin\lambda(x-s)\,d\lambda\right]ds. \tag{3}$$

We transform the kernel of expression (3) as follows:

$$G_{\bar{h}\bar{h}_0}(x-s) = -\mathrm{Re}\,i\int\limits_0^\infty e^{\lambda[-4(\bar{h}_0-\bar{h})-i(x-s)]}d\lambda -$$
$$-2\mathrm{Re}\,i\int\limits_0^\infty\frac{e^{-2\lambda\bar{h}_0}\,\mathrm{sh}^2\,2\lambda(\bar{h}_0-\bar{h})}{\mathrm{sh}\,2\lambda\bar{h}_0}e^{-i\lambda(x-s)}d\lambda = \tag{4}$$
$$= -\mathrm{Re}\,i\int\limits_0^\infty D(\lambda,\bar{h},\bar{h}_0)\,e^{-i\lambda(x-s)}d\lambda,$$

where

$$D(\lambda,\bar{h},\bar{h}_0) = e^{-4(\bar{h}_0-\bar{h})\lambda} + 2\,\frac{e^{-2\lambda\bar{h}_0}\,\mathrm{sh}\,2\lambda(\bar{h}_0-\bar{h})}{\mathrm{sh}\,2\lambda\bar{h}_0}. \tag{5}$$

We may then write equation (3) as follows:

$$L_1\gamma = -\frac{\mathrm{Re}}{2\pi}i\int\limits_0^\infty D(\lambda,\bar{h}_0,\bar{h})\,e^{-i\lambda x}H_1(\lambda)\,d\lambda, \tag{6}$$

where $H_1(\lambda)$ is N. Ye. Kochin's function,

$$H_1(\lambda) = \int\limits_{-1}^{+1}\gamma_1(s)\,e^{i\lambda s}ds.$$

Using the operator L_0^{-1} in the class of functions c_0^{∞} and integrating in the interval from -1 to +1, we get from expressions (2) and (6) /104

$$\int_{-1}^{+1} \gamma_1(x)\,dx = \int_{-1}^{+1} \gamma_0(x)\,dx - j\int_0^{\infty} D(\lambda, \bar{h}_0, \bar{h})\,H(\lambda)\,P(\lambda)\,d\lambda;$$

$$P(\lambda) = -\frac{1}{\pi^2}\int_{-1}^{+1} \sqrt{\frac{1+x}{1-x}}\int_{-1}^{+1} \sqrt{\frac{1-s}{1+s}}\,\frac{e^{-j\lambda s}}{x-s}\,ds.$$

The expression for the influence function of the deflector $\psi = C_y/C_{y\infty}$ determined in the first approximation by using Kochin's functions takes the form [3]

$$\psi = \frac{1}{1 + j\dfrac{\int_0^{\infty} D(\lambda, \bar{h}_0, \bar{h})\,H_0(\lambda)\,P(\lambda)\,d\lambda}{H_0(0)}}.$$

After being determined

$$H_0(\lambda) = \int_{-1}^{+1} \gamma_{10}(x)\,e^{j\lambda x}dx;$$

$$H_0(0) = \int_{-1}^{+1} \gamma_{10}(x)\,dx.$$

Here $\gamma_{10}(x)$ is the distribution function taken from the problem for an infinite fluid [3].

$$\gamma_{10}(x) = \frac{2}{\pi}\sqrt{\frac{1+x}{1-x}}\int_{-1}^{+1} \sqrt{\frac{1-s}{1+s}}\,\frac{\varphi_2}{s-x}\,ds +$$

$$+ ik\left[C + \frac{\pi i}{2} + \ln k\right]A_1\sqrt{\frac{1+x}{1-x}} - 8ik\,\operatorname{arctg}\sqrt{\frac{1+x}{1-x}}\cdot A_1,$$

where

$$A_1 = \int_{-1}^{+1} \sqrt{\frac{1-s}{1+s}}\,\varphi_2\,ds;$$

C is Euler's constant; ϕ_z is the derivative of the velocity potential.

By way of example, let us consider translation vibrations of the wing. Then $\phi_z = 1$, $A_1 = \pi$.

Leaving out intermediate steps, we write the expression for ψ in the following form:

$$\psi = \frac{1}{1 + \frac{A_{01}}{B - 2ik}},$$

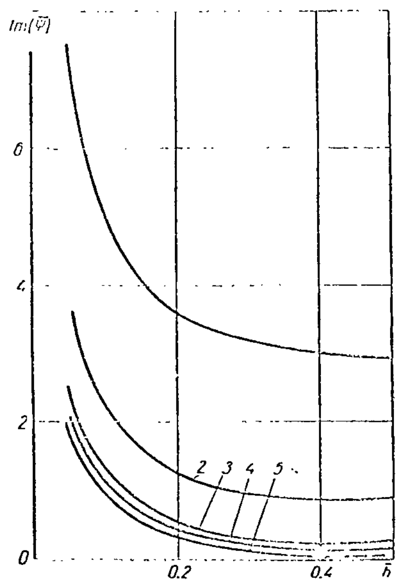
where /105

$$A_{01} = -2B\int_0^\infty D(\lambda, \bar{h}_0, \bar{h})\, I_0 I_1 d\lambda + 4ik\int_0^\infty D(\lambda, \bar{h}_0, \bar{h})\, \frac{I_1^2}{\lambda}\, d\lambda;$$

$$B = 1 + \frac{ik}{\pi}\left(C + \frac{\pi i}{2} + \ln k\right) \qquad (7)$$

($I_0(\lambda)$ and $I_1(\lambda)$ are Bessel functions).

It is shown in [1, 2] that a correspondence may be established between the functions $G_{\overline{hh}_0}(x - s)$ for a liquid of finite depth and $G_{\bar{h}}(x - s)$ for a fluid of infinite depth, and that the final result may be derived by appropriate substitutions in the solutions.



Curves for the Function $\mathrm{Im}(\bar{\psi}) = f(\bar{h}, \bar{h}_1)$ when $k = 0.05$: 1, $\bar{h}_1 = 0.05$; 2, $\bar{h}_1 = 0.1$; 3, $\bar{h}_1 = 0.2$; 4, $\bar{h}_1 = 0.3$; 5, $\bar{h}_1 = 0.5$.

Using this circumstance, the integrals which appear in expression (7) may be represented in the form of series with respect to the parameter ε which accounts for the finite depth of the liquid rather than in the form of series with respect to τ (as is done in [3]). Extensive tables are given in [1] for ε as a function of $\bar{h}$, $\bar{h}_0$ and the Froude number.

Then

$$\int_0 D(\lambda, \bar{h}_0, \bar{h})\, I_0 I_1 d\lambda = -\frac{1}{2}\varepsilon_1 + \frac{1}{8}\varepsilon_2 - \frac{1}{8}\varepsilon_3 + \ldots;$$

$$\int_0^\infty D(\lambda, \bar{h}_0, \bar{h})\, \frac{I_1^2}{\lambda}\, d\lambda = -\frac{1}{2}\varepsilon_1 - \frac{1}{4}\varepsilon_2 - 0{,}0625\varepsilon_3 + \ldots$$

The final formula for ψ will take the form

$$\psi = \frac{1}{1+\varepsilon_1 - \frac{1}{4}\varepsilon_2 + \frac{1}{4}\varepsilon_3} + i\,\frac{k}{1-k}\cdot\frac{-4\varepsilon_1 + 1{,}5\varepsilon_2 - 0{,}25\varepsilon_3}{\left(1+\varepsilon_1 - \frac{1}{4}\varepsilon_2 + \frac{1}{4}\varepsilon_3\right)^2}.$$

Curves for the imaginary part of the function $\bar{\psi}$ (see figure) show that the effect of variability in the process decreases considerably with an increase in relative distances $\bar{h}$ and $\bar{h}_1$, where $\bar{h}_1 = \bar{h}_0 - \bar{h}$.

The function $\bar{\psi}$ takes the form [3]

$$\bar{\psi} = 1 + i\,\frac{k}{1-k}\cdot\frac{-4\varepsilon_1 + 1{,}5\varepsilon_2 - 0{,}25\varepsilon_3}{1+\varepsilon_1 - \frac{1}{4}\varepsilon_2 + \frac{1}{4}\varepsilon_3}.$$

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CRITERION FOR AUTOSTABILIZATION OF THE MOTION OF A SYSTEM OF TWO WINGS CLOSE TO A DEFLECTOR

A. N. Golubentsev, A. P. Akimenko and N. F. Kirichenko

ABSTRACT: Parameters are found for a system of two rigidly interconnected wings such that the motion of the system is maintained in a state of stability or asymptotic stability when subjected to perturbations which are given in a finite region. The problem is solved in the nonlinear formulation.

A system of two rigidly connected wings moves at a constant horizontal velocity close to the interface between two media (Figure 1). The ox-axis lies on the interface between the two media; the oy-axis moves together with the system and passes through the center of gravity of the system at every moment. /107

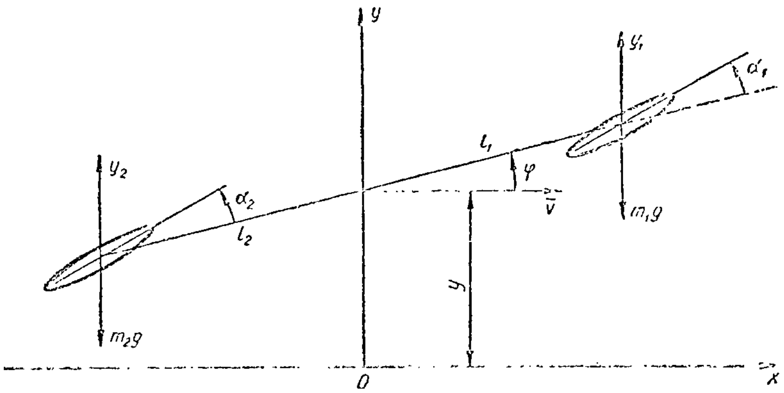

Fig. 1. Schematic Diagram of the System of Two Rigidly Connected Wings.

The centers of gravity of the first and second wings lie on the lines of action of the corresponding lift forces I_1 and I_2 (α_1 and α_2 are the angles between the chords of the first and second wings and the axis of the vehicle).

The equations of motion of the system with regard to the effect of change in the angles of inclination of the chords of the wings to the ox-axis and disregarding the change in vertical displacements in disturbed motion take the form

$$m\ddot{y} = [C_y(\lambda_1, \varphi + \alpha_1, y + l_1 \sin\varphi) S_1 + C_y(\lambda_2, \varphi + \alpha_2, y + l_2 \sin\varphi) S_2] \frac{\varrho v^2}{2} - mg - k_1 \dot{y};$$

$$I\ddot{\varphi} = [C_y(\lambda_1, \varphi + \alpha_1, y + l_1 \sin\varphi) S_1 l_1 - C_y(\lambda_2, \varphi + \alpha_2, y - l_2 \sin\varphi) S_2 l_2 + m_z(\lambda_1, \varphi + \alpha_1, y + l_1 \sin\varphi) b_1 S_1 + m_z(\lambda_2, \varphi + \alpha_2, y - l_2 \sin\varphi) b_2 S_2] \frac{\varrho v^2}{2} + m_2 g l_2 - m_1 g l_1 - k_2 \dot{\varphi}, \quad (1)$$

where λ_1 and λ_2 are the aspect ratios of the first and second wings, respectively; C_y is the coefficient of lift of the wing; k_1 and k_2 are the damping coefficients of the medium.

The equilibrium position of the system may be determined from the following system of equations [1]:

$$C_y(\lambda_1, \alpha_1, h_0) S_1 \frac{\varrho v^2}{2} + C_y(\lambda_2, \alpha_2, h_0) S_2 \frac{\varrho v^2}{2} = mg;$$

$$C_y(\lambda_1, \alpha_1, h_0) S_1 l_1 - C_y(\lambda_2, \alpha_2, h_0) S_2 l_2 + m_z(\lambda_1, \alpha_1, h_0) b_1 S_1 + m_z(\lambda_2, \alpha_2, h_0) b_2 S_2 = (m_1 g l_1 + m_2 g l_2) \frac{2}{\varrho v^2};$$

$$\varphi = 0. \quad (2)$$ /108

The equations of disturbed motion are then written in the form

$$\Delta\ddot{y} = A_1 \Delta y + A_2 \Delta\varphi + A_3 \Delta\varphi \Delta y + A_4 \Delta\varphi^3 + A_5 \Delta\varphi \Delta y^2 + A_6 \Delta\varphi^2 + A_7 \Delta y^2 - \frac{k_1}{m} \Delta\dot{y}^2;$$

$$\Delta\ddot{\varphi} = B_1 \Delta\varphi + B_2 \Delta y + B_3 \Delta\varphi \Delta y + B_4 \Delta\varphi^3 + B_5 \Delta\varphi \Delta y^2 + B_6 \Delta\varphi^2 + B_7 \Delta y^2 - \frac{k_2}{I} \Delta\dot{\varphi}, \quad (3)$$

where

$$A_1 = \left[\frac{\partial C_y(\lambda_1, \alpha_1, y)}{\partial y}\Big|_{y=h_0} S_1 + \frac{\partial C_y(\lambda_2, \alpha_2, y)}{\partial y}\Big|_{y=h_0} S_2 \right] \frac{\varrho v^2}{2m};$$

$$A_2 = \sum_{i=1}^{2} \left[\frac{\partial^2 C_y(\lambda_i, \alpha_i, y)}{\partial y^2}\Big|_{y=h_0} l_i S_i (-1)^{i-1} + \frac{\partial C_y(\lambda_i, \varphi + \alpha_i, h_0)}{\partial \varphi}\Big|_{\varphi=0} S_i \right] \frac{\varrho v^2}{2m};$$

$$A_3 = \sum_{i=1}^{2} \left[\frac{\partial^2 C_y(\lambda_i, \alpha_i, y)}{\partial y^2}\Big|_{y=h_0} l_i S_i (-1)^{i-1} + \frac{\partial^2 C_y(\lambda_i, \varphi + \alpha_i, y)}{\partial \varphi \partial y}\Big|_{\substack{\varphi=0\\ y=h_0}} S_1 \right] \frac{\varrho v^2}{2m};$$

/109

$$A_4 = \frac{1}{2} \sum_{i=1}^{2} \frac{\partial^3 C_y(\lambda_i, \varphi + \alpha_i, y)}{\partial \varphi \partial y^2}\Big|_{\substack{\varphi=0\\ y=h_0}} S_i l_i^2 \frac{\varrho v^2}{2m};$$

$$A_5 = \frac{1}{2} \sum_{i=1}^{2} \frac{\partial^3 C_y(\lambda_i, \varphi + \alpha_i, y)}{\partial \varphi \partial y^2}\Big|_{\substack{\varphi=0\\ y=h_0}} S_i \frac{1}{m};$$

$$A_6 = \frac{1}{2} \sum_{i=1}^{2} \frac{\partial^2 C_y(\lambda_i, \alpha_i, y)}{\partial y^2}\Big|_{y=h_0} l_i^2 S_i \frac{1}{m};$$

$$A_7 = \frac{1}{2} \sum_{i=1}^{2} \frac{\partial^2 C_y(\lambda_i, \alpha_i, y)}{\partial y^2}\Big|_{y=h_0} S_i \frac{1}{m};$$

$$B_1 = \sum_{i=1}^{2} \left[\frac{\partial C_y(\lambda_i, \alpha_i, y)}{\partial y}\Big|_{y=h_0} S_i l_i^2 + (-1)^{i-1} \frac{\partial C_y(\lambda_i, \varphi + \alpha_i, h_0)}{\partial \varphi}\Big|_{\varphi=0} S_i l_i + \frac{\partial m_z(\lambda_i, \alpha_i, y)}{\partial y}\Big|_{y=h_0} S_i b_i l_i (-1)^{i-1} + \frac{\partial m_z(\lambda_i, \varphi + \alpha_i, h_0)}{\partial \varphi}\Big|_{\varphi=0} S_i b_i \right] \frac{\varrho v^2}{2I};$$

$$B_2 = \sum_{i=1}^{2} \left[\frac{\partial C_y(\lambda_i, \alpha_i, y)}{\partial y}\Big|_{y=h_0} S_i (-1)^{i-1} l_i + \frac{\partial m_z(\lambda_i, \alpha_i, y)}{\partial y}\Big|_{y=h_0} b_i S_i \right] \frac{\varrho v^2}{2I};$$

$$B_3 = \sum_{i=1}^{2} \left[\frac{\partial^2 C_y(\lambda_i, \alpha_i, y)}{\partial y^2}\Big|_{y=h_0} l_i^2 S_i + \frac{\partial^2 C_y(\lambda_i, \varphi + \alpha_i, y)}{\partial \varphi \partial y}\Big|_{\substack{\varphi=0\\ y=h_0}} S_i l_i + \frac{\partial^2 m_z(\lambda_i, \alpha_i, y)}{\partial y^2}\Big|_{y=h_0} l_i b_i S_i (-1)^{i-1} + \frac{\partial^2 m_z(\lambda_i, \varphi + \alpha_i, y)}{\partial \varphi \partial y}\Big|_{\substack{\varphi=0\\ y=h_0}} b_i S_i \right] \frac{\varrho v^2}{2I};$$

$$B_4 = \frac{1}{2} \sum_{i=1}^{2} \left[\frac{\partial^3 C_y(\lambda_i, \varphi + \alpha_i, y)}{\partial \varphi \partial y^2}\Big|_{\substack{\varphi=0\\ y=h_0}} S_i l_i^3 (-1)^{i-1} + \right.$$

$$+\frac{\partial^3 m_z(\lambda_i, \varphi+\alpha_i, y)}{\partial\varphi\partial y^2}\Big|_{\substack{\varphi=0\\y=h_0}} b_i S_i l_i^2\Big]\frac{\rho v^2}{2I};$$

$$B_5 = \frac{1}{2}\sum_{i=1}\Big[\frac{\partial C^3_y(\lambda_i, \varphi+\alpha_i, y)}{\partial\varphi\partial y^2}\Big|_{\substack{\varphi=0\\y=h_0}} S_i l_i(-1)^{i-1} +$$

$$+\frac{\partial^3 m_z(\lambda_i, \varphi+\alpha_i, y)}{\partial\varphi\partial y^2}\Big|_{\substack{\varphi=0\\y=h_0}} S_i b_i\Big]\frac{\rho v^2}{2I};$$

/110

$$B_6 = \frac{1}{2}\sum_{i=1}^{2}\Big[\frac{\partial^2 C_y(\lambda_i, \alpha_i, y)}{\partial y^2}\Big|_{y=h_0} l_i^3 S_i(-1)^{i-1} +$$

$$+\frac{\partial^2 m_z(\lambda_i, \alpha_i, y)}{\partial y^2}\Big|_{y=h_0} l_i^2 S_i b_i\Big]\frac{\rho v^2}{2I};$$

$$B_7 = \frac{1}{2}\sum_{i=1}^{2}\Big[\frac{\partial^2 C_y(\lambda_i, \alpha_i, y)}{\partial y^2}\Big|_{y=h_0} l_i S_i(-1)^{i-1} +$$

$$+\frac{\partial^2 m_z(\lambda_i, \alpha_i, y)}{\partial y^2}\Big|_{y=h_0} b_i S_i\Big]\frac{\rho v^2}{2I}.$$

Let us find the conditions which must be satisfied by the parameters of the given system so that the requirement

$$\left.\begin{aligned}
&|\varphi(t)| < \bar{\varphi};\\
&|\Delta y(t)| < h_0 - \bar{l}\sin\bar{\varphi} = \bar{y};\\
&\lim_{t\to\infty}\Delta y(t) \to 0;\\
&\lim_{t\to\infty}\Delta\varphi(t) \to 0.
\end{aligned}\right\} \qquad (4)$$

is satisfied for the coordinates of disturbed motion.

The physical meaning of the conditions in system (4) is that the vehicle does not collide with the interface and that asymptotic stability of disturbed motion is maintained.

Let us write a system (3) in normal form. For the sake of convenience, we shall use the following notation for the coefficients A_n and B_n:

$$\begin{aligned}
&\dot{x}_1 = x_2;\\
&\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + A_{21}x_1^2 + A_{23}x_3^2 + A_{24}x_3^3 +\\
&\qquad + A_{25}x_3x_1 + A_{26}x_3x_1^2;
\end{aligned} \qquad (5)$$

$$\begin{aligned} \dot{x}_3 &= x_4; \\ \dot{x}_4 &= a_{41}x_1 + a_{43}x_3 + a_{44}x_4 + A_{41}x_1^2 + A_{43}x_3^2 + A_{44}x_3^3 + \\ &\quad + A_{45}x_3x_1 + A_{46}x_3x_1^2. \end{aligned} \tag{5}$$

The necessary and sufficient conditions for asymptotic stability are readily established for the linear approximation of system (3):

$$\begin{gathered} a_{21} < 0; \qquad a_{43} < 0; \\ a_{21},\, a_{43} - a_{23}a_{11} > 0. \end{gathered} \tag{6}$$

Let us use N. G. Chetayev's method [3] to construct a Lyapunov function /111
in positively defined quadratic form:

$$v(x) = \sum_{i,j=1}^{4} C_{ij}x_ix_j. \tag{7}$$

The coefficients C_{ij} are determined from the condition

$$2\sum_{i,j=1}^{4} C_{ij}x_j \sum_{k=1}^{4} a_{ik}x_k = -\sum_{i=1}^{4} x_i^2 \tag{8}$$

or from the system of linear algebraic equations

$$2\sum_{i=1}^{4} C_{ij}a_{ik} = -\delta_{jk} \qquad (j,\, k = 1,\, 2,\, 3,\, 4). \tag{9}$$

Here $a_{ik} = 0$ for those subscripts which do not appear in system (5); δ_{ij} is the Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & i \neq j; \\ 1, & i = j. \end{cases}$$

Let us determine the value of the coefficient C for which the set

$$\{x : v(x) = C\}$$

satisfies the condition

$$\{x : v(x) = C\} \subset \{x : |x_1| \leqslant \bar{y}; \quad |x_3| \leqslant \bar{\varphi}\}. \tag{10}$$

This value of C is determined as follows [2]:

$$C = \min \begin{cases} \dfrac{\bar{y}^2 A_4}{M_1}\,; \\ \dfrac{\bar{\varphi}^2 A_4}{M_3}\,, \end{cases} \tag{11}$$

where $A_4 = \det ||C_{ij}||$; M_k are the complementary minors to the k-th diagonal element of the matrix.

When the inequality

$$\left(\frac{dv}{dt}\right)_{(5)} < 0 \tag{12}$$

is satisfied in the region

$$\begin{aligned} &|x_1| < \bar{y} = \bar{x}_1; && |x_3| < \bar{\varphi} = \bar{x}_3; \\ &|x_2| \leqslant \sqrt{\frac{CM_2}{A_4}} = \bar{x}_2; && |x_4| \leqslant \sqrt{\frac{CM_4}{A_4}} = \bar{x}_4 \end{aligned} \tag{13}$$

then requirement (4) is fulfilled if the perturbation at the initial instant /112
satisfies the condition

$$\sum_{i,j=1}^{4} C_{ij} x_i(0)\, x_j(0) \leqslant C. \tag{14}$$

Condition (12) takes the form

$$\begin{gathered}
-\sum_{i=1}^{4} x_i^2 + 2\sum_{i,j=1}^{4} C_{ij} x_i \Phi_j(x) = -\sum_{i=1}^{4} x_i^2 + 2\sum_{i=1}^{4} x_i\, [x_i^3 (C_{i2}A_{21} + C_{i4}A_{41}) + \\
+ x_3^2 (C_{i2}A_{23} + C_{i4}A_{43}) + x_3^3 (C_{i2}A_{24} + C_{i4}A_{44}) + x_3 x_1 (C_{i2}A_{25} + C_{i4}A_{45}) + \\
+ x_3 x_1^2 (C_{i2}A_{23} + C_{i4}A_{44})] \leqslant \sum_{i=1}^{4} \bar{x}_i \{2\,|C_{i2}A_{21} + C_{i4}A_{41}| + 2\,|C_{i2}A_{23} + \\
+ C_{i4}A_{43}| + 2\bar{x}_3\,|C_{i2}A_{24} + C_{i4}A_{44}| + |C_{i2}A_{25} + C_{i4}A_{45}| + \\
+ 2\bar{x}_3\,|C_{i2}A_{36} + C_{i4}A_{46}| - 1\} \sum_{i=1}^{4} x_i^2 < 0 \qquad (i,\, j = 1, 2, 3, 4)
\end{gathered} \tag{15}$$

when x $\neq$ 0, where

$$\begin{aligned}
&\Phi_1 = 0; \qquad \Phi_2(x) = A_{21}x_1^2 + A_{23}x_3^2 + A_{24}x_3^3 + A_{25}x_3x_1 + A_{26}x_3x_1^2; \\
&\Phi_3 = 0; \qquad \Phi_4(x) = A_{41}x_1^2 + A_{43}x_3^2 + A_{44}x_3^3 + A_{45}x_3x_1 + A_{46}x_3x_1^2.
\end{aligned}$$

Inequality (15) is satisfied following the system

$$\begin{gathered}
2\,|C_{i2}A_{21} + C_{i4}A_{41}| + 2\,|C_{i2}A_{23} + C_{i4}A_{43}| + |C_{i2}A_{25} + C_{i4}A_{45}| + \\
+ 2\bar{x}_3 (|C_{i2}A_{24} + C_{i4}A_{44}| + |C_{i2}A_{26} + C_{i4}A_{46}|) \leqslant \frac{1}{4\bar{x}_i}.
\end{gathered} \tag{16}$$

Consequently, if the parameters of the system satisfy conditions (16), then the system is autostabilized in the sense of relationships (4) for perturbations bounded by the region of values (14).

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ON THE FRANKEL PROBLEM FOR CASCADE BLADING

G. S. Lipovoy

ABSTRACT: The author discusses formulation of the problem for near sonic flow around a vane cascade assuming compression shocks.

The problem for near sonic flow around an isolated blade where there is a shock wave is outlined in [4]. Formulation of the analogous problem for cascade blading is considered below. /113

It is known [1,3] that at gas flows greater than a certain critical value M_{cr_1}, cascade blading shows a series of compression shocks with a fairly complex structure which varies with a change in the mach number M. At certain values of M, the compression shocks have a finite amplitude, but they may become infinite with an increase in M. In the following discussion we shall limit ourselves to flows with finite compression shocks (Figure 1). For the sake of simplicity, we shall further assume that the compression shocks are rectilinear.

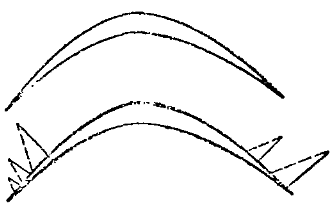

Fig. 1.

In order to explain the singularities of the stream function ψ in near sonic flow, let us first consider vortex flow of an incompressible fluid around the cascade. The complex potential and associated flow velocity [3] take the following form:

$$W = \bar{v}_\infty z + \frac{\Gamma}{2\pi i} \ln \operatorname{sh} \frac{\pi}{t} z + \text{const}; \tag{1}$$

$$\bar{v} = \bar{v}_\infty + \frac{\Gamma}{2\pi i} \operatorname{cth} \frac{\pi}{t} z. \tag{2}$$

Using the notation

$$\bar{v}_\infty - \frac{\Gamma}{2it} = \bar{v}_1; \quad \bar{v}_\infty + \frac{\Gamma}{2it} = \bar{v}_2,$$

we get /114

$$z = \frac{t}{2\pi} \ln \frac{\bar{v} - \bar{v}_1}{\bar{v} - \bar{v}_2}. \tag{3}$$

Then the expression for the complex potential in the hodograph plane will be

$$W = \frac{\bar{v}_1 t}{2\pi} \ln(\bar{v} - \bar{v}_1) - \frac{t}{2\pi}\left(\bar{v}_1 + \frac{i\Gamma}{t}\right) \ln(\bar{v} - \bar{v}_2) + \text{const.} \tag{4}$$

In the general case, the complex potential in the hodograph plane is

$$W = \frac{\Gamma_1 + iQ_1}{2\pi i} \ln(\bar{v} - \bar{v}_1) + \frac{\Gamma_2 + iQ_2}{2\pi i} \ln(\bar{v} - \bar{v}_2) + F(v), \tag{5}$$

where $Q_{1,2}$ is the flow rate; $\Gamma_{1,2}$ are the components of circulation. The function F(v) has no discontinuities within the hodograph. Using the notations $\bar{v} = \bar{v}_1 = \rho_1 e^{-iw_1}$, $\bar{v} - \bar{v}_2 = \rho_2 e^{-iw_2}$, the stream function may be represented as

$$\psi = \alpha \ln \varrho_1 + \beta \ln \varrho_2 + 0(\varrho). \tag{6}$$

The function ψ will have the same singularities in a gas flow. Let us recall that the region of the hodograph has an infinite number of sheets. As is conventional, let us consider only the part of this region corresponding to a single step of the cascade. For the sake of simplicity, we shall assume that this part of the region is one-sheeted, i.e. this region does not contain a point v = 0 and branching points. Then let us examine gas flow. We shall assume that the function ψ satisfies the equation (*Trikomi-Fal'kovich* gas)

$$\eta \frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial\eta^2} = 0, \tag{7}$$

where η is the known function for the modulus of velocity v;

$$\eta = \left(\frac{3}{2}\int_v^{a_*} \frac{\sqrt{a^2 - v^2}}{av}\, dv\right)^{\frac{2}{3}};$$

θ is the angle of inclination of the velocity vector.

We assume that the velocity potential ϕ exists. The problem is to find a solution for equation (7) in region D (Fig. 2) with singularities of type (6) under the following boundary conditions:

$$\psi = 0 \quad \text{or} \quad AG, a_2B, b_2C, \ldots; \tag{8}$$

$$\frac{\partial\psi}{\partial\theta} = 0 \quad \text{or} \quad AaA', BbB', \ldots; \tag{9}$$

$$\psi(\theta_n, \eta) = \psi(\theta_n, -\eta), (\theta_n = a, b, c, \ldots); \tag{10}$$

$$\psi_n(\theta, 0) = \tau_n(\theta) \quad \text{or} \quad a_1a_2, b_1b_2, \ldots, \tag{11}$$ /115

where $\tau_n(\theta)$ are given functions; the hodographs of arcs AG, a_2B, b_2C are given, and the hodographs of arcs $A'a_2$, $B'b_2$, $C'c_2$... are found during the process of solution as the locus of points of the hyperbolic part of the plane analogously to the problem for an isolated blade [4], where $\psi = 0$. Points a_1, b_1, c_1... are the points of intersection of the characteristics passing through the points A', B', C'... with the Oη-axis. The singularities of the function ψ are located at the points (θ_1, η_1), (θ_2, η_2), corresponding to the values of velocity before the cascade and after the cascade.

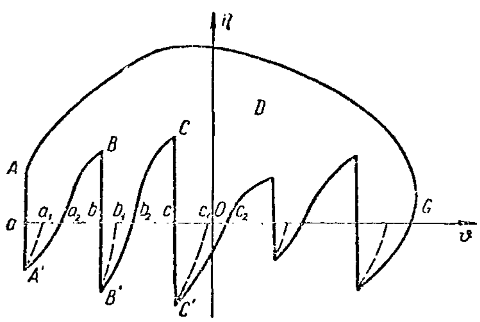

Fig. 2.

Let us compute the values of the constants α and β appearing in equation (6). In order to do this, we make the transition from the plane θ, η to the θ, s (where

$$s = \frac{2}{3}\eta^{\frac{3}{2}})$$ and introduce polar coordinates

$$\theta - \theta_{1,2} = \varrho_{1,2}\sin\omega_{1,2}; \quad s - s_{1,2} = \varrho_{1,2}\cos\omega_{1,2}.$$

in the neighborhoods of singular points.

Using the relationship between ϕ and ψ according to S. V. Fal'kovich's equation:

$$\frac{\partial\varphi}{\partial\varrho} = C\left(\frac{3}{2}s\right)^{1/3}\frac{\partial\psi}{\varrho\partial\omega},\quad \frac{\partial\varphi}{\varrho\partial\omega} = -C\left(\frac{3}{2}s\right)^{1/3}\frac{\partial\psi}{\partial\varrho};$$

$$\left[C = \left(\frac{\chi+1}{2}\right)^{1/\chi-1}(\chi+1)^{1/3}\right], \qquad (12)$$

as well as the relationships

$$\oint d\varphi = \Gamma;\quad \oint\frac{e^{i\theta}}{v}\left(d\varphi + i\frac{\sigma_0}{\sigma}d\psi\right) = 0 \qquad (13)$$

(where the contour of integration should include both singular points), it may be shown that /116

$$\alpha = \frac{v_1\Gamma}{c2\pi\left(\frac{3}{2}s_1\right)^{1/3}(v_2 - v_1)};\quad \beta = \frac{v_2\Gamma}{c2\pi\left(\frac{3}{2}s_2\right)^{1/3}(v_1 - v_2)}. \qquad (14)$$

This problem may be solved by using the following method of successive approximations. When $\tau_n(\theta) = \psi_n(\theta, 0)$ are given on segments aa_1, bb_1, cc_1, ...we find the first approximation $\psi^{(1)}$ in the elliptical part of the region. Then we use equations (9) and (10) to solve the Kauchy problem in the given region: $aA'a_2$, $bB'b_2$, $cC'c_2$.... As a result, we learn the value of ψ on the characteristics which pass through points a, b, c,... Having these quantities, and using the value of $\frac{\partial\psi}{\partial n}^{(1)}$, determined earlier, we get new values for $\tau_n(\theta)$. The questions of the existence and uniqueness of the solution remain to be studied.

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CHEMICAL COATING METHOD FOR MAKING THE LAMINAR SECTION OF A BOUNDARY LAYER

L. F. Kozlov and A. F. Mozhanskaya

ABSTRACT: A detailed description is given of a technique for making flow in a boundary layer visible by a chemical coating method when testing models in an aqueous medium.

The chemical coating method is based on the difference in the erosion intensity for a chemical applied to the surface of a model in laminar and turbulent flows. Experience has shown that this method is extremely simple and its results are reliable. However, the given method has not been extensively used, apparently because of the lack of a detailed description of the technique involved. /117

Biacetyl of hydroquinone $C_{10}H_{10}O_4$ which is a complex ester of diatomic phenol is recommended in the case where the chemical coating method is used as an indicator. Biacetyl of hydroquinone in pure form is a colorless, odorless crystal which is insoluble in water under ordinary conditions and dissolves in acetone in any concentrations. It is synthesized by esterification reaction between hydroquinone $C_6H_4(OH)_2$ and aceticanhydride $(CH_3CO)_2O$.

Acetic acid is formed first during the reaction:

$$(CH_3CO)_2O + H_2O \rightarrow 2\,CH_3COOH,$$

which reacts with hydroquinone to give biacetyl of hydroquinone:

$$2\,CH_3COOH + C_6H_4(OH)_2 \rightleftarrows C_{10}H_{10}O_4 + 2H_2O.$$

The reaction takes place in the presence of sulfuric acid which acts as a catalyst.

Biacetyl of hydroquinone is prepared by adding two parts by weight of acetic anhydride and one or two drops of concentrated sulfuric acid to one part by weight of hydroquinone. The reaction is exothermic and takes place vigorously and therefore should be carried out under an exhaust hood with constant agitation of the mixture. The hydroquinone is completely dissolved by the reaction, and the beaker in which the reaction takes place is filled with a hot dark yellow liquid with the acrid odor of acetic acid. Upon completion of the reaction, the solution begins to congeal, forming a thick gray crystalline mass. After solidifying the crystalline mass is thoroughly washed in water until the acetic acid disappears. The washed crystals are dried in a drying cabinet or in the air. /118

If the initial product were insufficiently pure, the resultant crystals of biacetyl of hydroquinone have a grayish tinge. In this case, the resultant biacetyl of hydroquinone should be recrystallized. For this purpose, small portions of the compound (10-20 g) are mixed in a chemical beaker with a capacity of approximately 500 cm^3, covered with water and boiled. The boiling solution is then quickly filtered through two or three layers of gauze and the undissolved crystals are boiled again. During solidification of the filtrate, crystals of biacetyl of hydroquinone settle on the bottom of the beaker. After drying, the biacetyl of hydroquinone recrystallized in this way has the form of small light snow-white crystals and is suitable for use.

The quality of the results of tests is dependent to a considerable extent on the state of the surface of the model. The model should be carefully made and should have a fifth-class surface finish (GOST 2789-59). The nose section of the model should be finished with special care. The surfaces of wooden and paraffin models are finished differently. Wooden models are coated with a black lacquer which adheres well to the wood, is resistant to water and acetone and forms a film which does not lose elasticity with age. The conventional standard codings (nitroglaze, oil-based varnishes and paints) are not suitable for this purpose since the films produced by these coatings are destroyed by acetone. The given requirements are satisfied by a coating with the following composition: VIAM-B-3 resin (100 wt. %), kerosene based Twitchell reagent (17 wt.%),solvent (3 wt.%) and plasticizer (10 wt.%).

Before application of the lacquer, the surface of the model is carefully filled and sanded. The filler is made of fine sawdust and VIAM-B-3 resin. If there are oil spots on the surface, they must be removed with a rag soaked in gasoline or acetone. The lacquer is prepared immediately before use. The solvent, Twitchell reagent and plasticizer are added to the resin one after the other. The lacquer is mixed after adding each component. The finished lacquer has a liquid consistency and is applied with the grain using a soft brush. The first coat is forced into the wood by strong pressure on the brush and is followed by three or four coats which are applied at intervals of 6-8 hours. Before application of the third and fourth layers, 3-5% /119
lamp black is added to the lacquer to give the coating a uniform black color. The lacquer is used at a rate of 70-90 g/m^2.

A paraffin model may be tested by this method without first coloring the surface. However, in this case the results of the tests are less distinct (especially on photographs) because of the lack of contrast between the color of the indicator coating and the color of the model surface. Therefore the surface of a paraffin model should be blackened 2-3 days before the beginning of tests using the following composition: 23% carbon black, 7% beeswax and 70% gasoline. The preparation is abundantly applied to the paraffin surface of the model and rubbed in with a rag until the paraffin is uniformly black in color. The surface is then dried for several hours and polished with a soft rag.

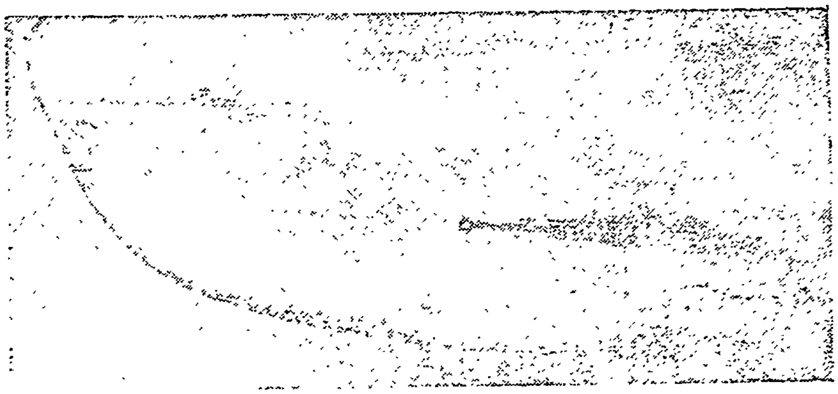

Fig. 1.

The indicator solution is prepared immediately before the beginning of tests--7 g of biacetyl of hydroquinone per 100 cc of commerical acetone. To prevent clogging of the atomizer during work, the solution should be filtered before use. The solution is used at the rate of about 100 cc per 1 m^2 of surface to be coated.

The surface of a model suspended above water must be washed with soap, scrubbed with a brush and wiped dry. When a wooden model is tested, the surface is coated with a thin layer of wax polish which is a 10% solution of beeswax in gasoline. A soft brush is used for applying the polish to the surface of the model which is then polished with a clean cloth. The model is studded with indicator dowel pegs having conical heads (the cones are 2-2.5 mm in diameter and 1.5-2 mm high) which make it easier to identify the boundaries of the laminar section in the boundary layer.

An ordinary paint sprayer is used for applying the indicator coating sol- /120
ution to the surface of the model. In applying the coating, the spray nozzle is moved uniformly at a distance of 60-70 cm from the surface of the model. Nearly dry crystals of biacetyl hydroquinone are deposited on the surface of the model, forming a whitish film. For experimental purposes, it is extremely important to produce a uniform biacetyl hydroquinone coating on the surface of the model. After application, the acetone is allowed to evaporate

for 2-3 minutes. The model is then carefully lowered into the water and towed. When the model is tested at a velocity of less than 1 m/sec, towing begins immediately after the model is secured, and at velocities of more than 1 m/sec, the model is held in the water long enough so that the total length of time in the water is equal to the time for a model being towed at a velocity of 1 m/sec. After towing, the model is raised out of the water and dried with an ordinary foehn dryer. Remaining on the dried surface is a white residue of indicator coating intersected by black cones (Fig. 1). The nuclei for these cones are the dowels or large crystals of indicator coating which act as centers of turbulent disturbances. It is recommended that a natural sponge be used for rubbing off large crystals of the chemical indicator coating to prevent distortion of the streamline flow before the model is lowered into the water.

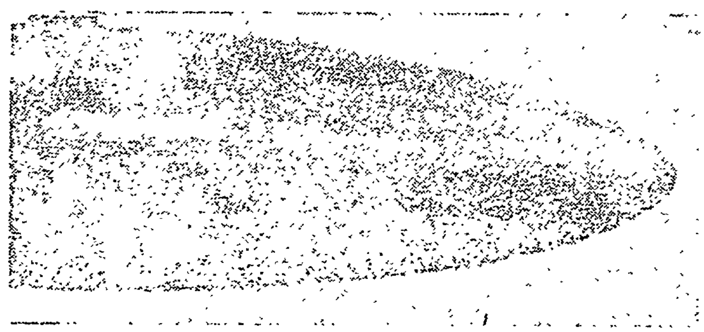

Fig. 2.

The region of the laminar section terminates with the part of the surface on which the black cones produced by the dowels blend with the general background of the surface. If the coating is applied too thickly or in a non-uniform layer, the white sections also remain in zones of transition and turbulent flow which are determined in these cases by the absence of cones behind the dowels. An indicator coating may also be used for highly accurate determination of wave profile. /121

The chemical coating method may be used for model basin tests of both trolley and gravity type models of the surface vessels and submarines. By way of example, Figures 2-3 show the results of tests on determining the laminar section in the boundary layer of a dirigible model tested in the underwater position at velocities of 1 and 4 m/sec respectively (the model was 1 meter long).

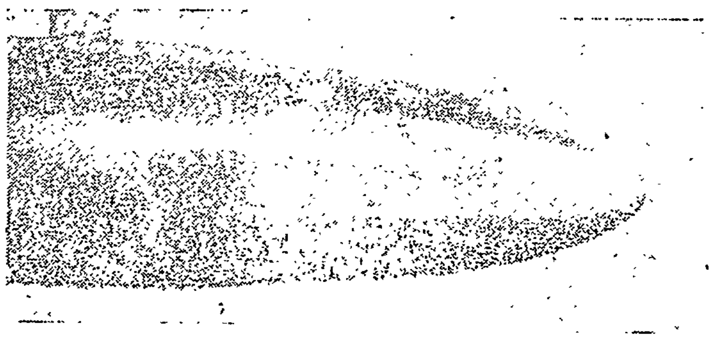

Fig. 3.

INVESTIGATION OF A VELOCITY PICKUP FOR MEASUREMENTS IN A WATER FLOW

A. V. Bugayenko and V. M. Shakalo

ABSTRACT: The authors consider two processes of electrical conductivity in a fluid where the amplitude of the electric current depends on flow velocity from the standpoint of using this velocity for measurement. Experimental measurements are described. It is shown that electrical conductivity may be used for measuring flow conditions.

The electrical conductivity of dissociating liquids depends upon flow /122 velocity [4]. This circumstance may be utilized for measuring the velocity and conditions of fluid flow. However, flow velocity may have various effects on the electrical conductivity of a fluid. The current flowing through an electrolyte is determined by three different processes in the electrolyte: the transfer of ions from within the solution to the surface of the electrode, electrochemical reaction and removal of the reaction product. The amplitude of the current depends chiefly on the rate of the slowest process.

When using a system of electrolyte and inert electrodes which provide a current due to the reaction rate, a relationship between current and flow rate may be derived on the basis of the reversibility of the phenomenon of electroosmosis [3]. In this case, as the electrolyte moves, opposing charges are induced on the electrodes which reduce the voltage by the quantity

$$E = \frac{2\rho U l}{r\varkappa}, \tag{1}$$

where r is the radius of the circular electrodes; ρ is the charge density in the double layer; κ is specific electrical conductivity; L is the distance between electrodes; U is the rate of laminar flow.

The rate of flow may also be determined by measuring the analogous flow potential in capillaries [2]. In the case of potential difference between the electrodes, the current density is expressed as

$$i = kce^{-\alpha F(\varepsilon - \gamma E)} \tag{2}$$

where k, α and γ are constants, c is the concentration of ions discharging at the electrodes; F is the Faraday constant.

Current in the electrolyte appears most frequently as the result of ion transfer which is due to three parallel processes: diffusion, convection and migration. The current density due to diffusion and migration is determined from the expression /123

$$i = \frac{kFz_1(z_2+z_1)c_0}{d}\left[D\exp\frac{Fz_1z_2u}{(z_1+z_2)RT} - D\right], \tag{3}$$

where D is the coefficient of diffusion for cations or anions, respectively, (assuming a binary electrolyte); c_0 is the concentration of ions in the solution; z_1, z_2 are the cation and anion charges; d is the distance between electrodes; k is a coefficient of proportionality which depends on the shape of the electrodes.

If the gradient of concentration at the cathode is higher than at the anode, the cofactor in brackets in formula (3) becomes 1.

During electrolyte motion, the convection current density is

$$i = \frac{DFz_1(z_1+z_2)c_0}{\delta}, \tag{4}$$

where δ is the thickness of the diffusion layer in which the concentration changes from the value c_0 in the solution to the value c_1 at the surface of the electrode, which is usually equal to 0 for the case of slow transfer.

The thickness of the diffusion layer depends on velocity and fluid flow conditions and differs for various electrode surfaces [1]. If flow measurement is done with electrodes in the form of large plates with surface area S located along the flow at a single distance x in an arbitrary coordinate system, then for laminar flow conditions

$$\delta = 3\left(\frac{D}{\nu}\right)^{\frac{1}{3}}\sqrt{\frac{\nu x}{U}}, \tag{5}$$

where ν is the coefficient of kinematic viscosity, and the convection current through the electrolyte is

$$I_k = \iint_S 3\left(\frac{D}{\nu}\right)^{\frac{1}{3}} \sqrt{\frac{\nu x}{U}}\, dS. \tag{6}$$

For a plate of length l and width b, we have

$$I_k = \mathrm{F} z_1 (z_1 + z_2)\, c_0 U^{\frac{1}{2}} \nu^{-\frac{1}{6}} D^{-\frac{1}{3}} b l^{\frac{1}{2}}. \tag{7}$$

When plates of smaller dimensions are used, the diffusion layer is so thin that the electrochemical reaction becomes the current-determining process. Relationship (2) will correspond to the current-voltage characteristics. An /124
increase in the voltage across the electrodes leads to an increase in E and consequently to a reduction in current.

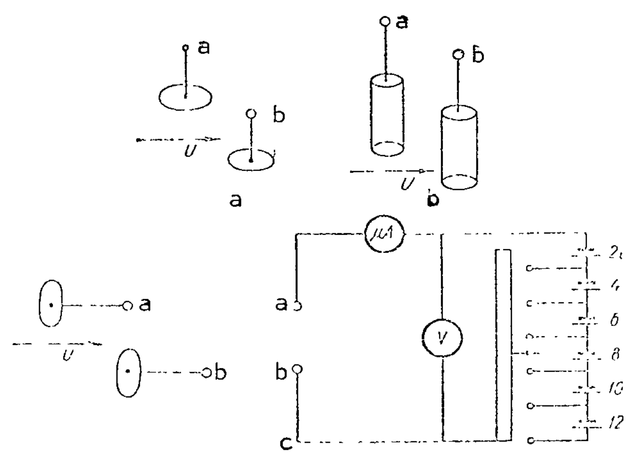

Fig. 1. Electrical Diagram for the Installation.

To check the hypothesis on current reduction with an increase in flow velocity and to determine the effect of relationships (2) and (4) on the current in the electrolyte, a special device (Fig. 1) was developed and installed in the flume at the Institute of Hydromechanics of the Academy of Sciences of the Ukrainian SSR for holding two electrodes in a stream of tap water with controllable flow velocity. The electrodes were designed so that the conducting surfaces could be completely immersed in the water. The seat of emf for the system was a sectional storage battery. The voltage was regulated by changing the number of battery cells. A pickup made by shearing off two insulated wires 0.8 mm in diameter located at a distance of 2 mm from each other was set with the shear surfaces (electrodes) parallel to the flow (Figure 1a). The ratio of the current through the moving electrolyte to that through a still electrolyte I/I_0 decreased with an increase in the rate of flow both for the case where the electrodes were set parallel to the flow and where they were set perpendicular to the flow (Figure 1c). In the latter case, there was a reduction in the slope of the current-voltage characteristics. In the case of electrodes made in the form of small cylinders 1.5 mm in height and 0.8 mm in diameter arranged as shown in Figure 1b, a rise in current was observed with an increase in velocity and at a voltage of 2v. At velocities in the range from 0.4 to 1.5 m/sec, the relationships are linear with the slope of the curve reaching 41 μa per m/sec. A rise in voltage in this case leads to a reduction in the slope of the curve (Figure 2). When U > 1.5 m/sec there is a noticeably sharp reduction in current due to cavitation near the electrodes. This phenomenon may be utilized for determining the beginning of flow cavitation. /125

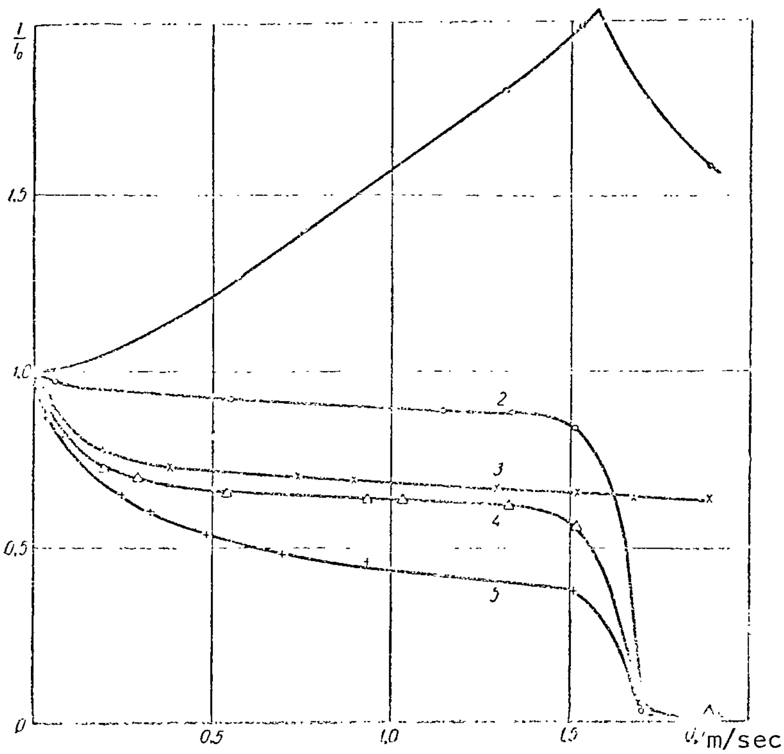

Fig. 2. Curves for Current Ratio as a Function of Flow Rate for Various Electrode Designs and Voltages: 1, U = 2 v (Fig. 1c); 2, U = 12 v (Fig. 1c); 3, U = 12 v (Fig. 1b); 4, U = 4 v (Fig. 1a); 5, U = 12 v (Fig. 1a).

When we compare expressions (2) and (3) for the case of cathodic concentration overvoltage, we come to the conclusion that the effect which temperature and constants have on measurement accuracy in a fluid which is flowing non-uniformly may be reduced to a minimum only for convection characteristics. The exact velocity is determined from dependence on the dimensionless quantity

$$\frac{I_k - I_{0k}}{I_{0k}}.$$

For laminar flow, the thickness of the diffusion layer is related to the thickness of the boundary layer δ_0 by the relationship [1] /126

$$\delta = 0,6\left(\frac{D}{\nu}\right)^{\frac{1}{3}}\delta_0. \tag{8}$$

Hence the thickness of the boundary layer may be determined from the amplitude of the convection current. If we take account of the fact that

$$\delta = \frac{0,7\mathrm{Pr}^{-\frac{3}{4}}D}{\sqrt{k_f}U}, \tag{9}$$

for a turbulent boundary layer, where Pr is the Prandtl diffusion number which is equal to 10^3 for water, and

$$\frac{1}{\sqrt{k_f}} = 4,1\lg(k_f\,\mathrm{Re}) + 1,7, \tag{10}$$

then the point of transition of the boundary layer from laminar to turbulent may be determined from the transition of the convection current relationship from a velocity with the power of 1/2 to velocity with a power of 1. For practical purposes, two plate electrodes must be cemented at the point to be studied on a non-conducting model for determining δ, or a series of electrode pairs must be used for determining the point of transition and measuring flow conditions.

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EFFICIENCY OF A SCREW TYPE GAS-HYDROJET VESSEL

Yu. G. Mokeyev and I. M. Chernyy

ABSTRACT: The authors describe various modifications for putting together a screw type gas-hydrojet installation and give a hydrodynamic analysis of a staged system from the standpoint of its efficiency and optimum distribution of the load between the stages for various forward velocities of the vehicle.

As is known, the thrust of a gas-hydrojet increases with the forward velocity just as in a ramjet engine. For all practical purposes, the thrust is low under starting conditions. For this reason, a vessel equipped with a gas-hydrojet engine will have unsatisfactory acceleration characteristics. /127

A rational installation is a combined vehicle with screw and gas-hydrojets which have mutually opposing characteristics in the qualitative sense. There are two possible modifications for combining screw and gas-hydrojets: with the propulsion units arranged in parallel or in stages (Figure 1). These modifications have their own advantages and disadvantages, particularly in the hydrodynamic sense. The staged combination is preferable for greater compactness, and also for the fact that the operation of the screw provides for considerable thrust by the gas-hydrojet even at low forward velocities since the pressure in the chamber p_c may reach values close to that required.

However, in both cases (all other things being equal) the question of the propulsive coefficient for the installations is not clear and requires appropriate analysis. This problem is rather complex in the case of real propulsion units.

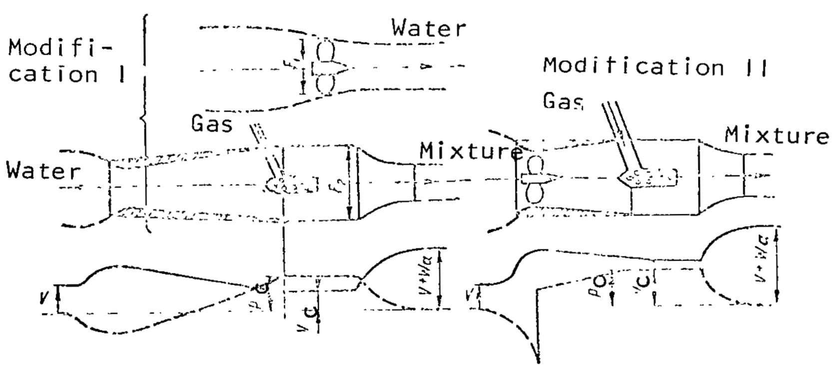

Fig. 1. Design Modifications for Combining a Screw and Gas-Hydrojet.

In the general case, to analyze and compare the efficiency of various combinations, we first consider three modifications of systems consisting of ideal propulsion units, i.e. active Froude discs:

a) First modification--two propulsion units acting in parallel with thrusts P_1 and P_2 and hydraulic cross-sections F_1 and F_2;

b) the two propulsion units replaced by a single unit which develops an over-all thrust $P = P_1 + P_2$ with over-all hydraulic cross-section $F = F_1 + F_2$;

c) two propulsion units which develop an over-all thrust P arranged in series, the hydraulic cross-section being equal to the cross-section of the larger engine, i.e. $F_1 = F$; $P = P_1 + P_2$

At first glance, the first two modifications appear to be equivalent, but this is only true when $P_1/F_1 = P_2/F_2$ or $\sigma_1 = \sigma_2$. Solution of the problem of analyzing the efficiency of installations has been fairly simple since identical propulsion units ($P_1 = P_2$; $F_1 = F_2$) are usually installed on vessels in practice. This problem is complicated for combination installations since the design values of σ_1 and σ_2 may be considerably different. /128

Let us proceed with the analysis of the efficiency of our given modifications for the assumed conditions. In the case of parallel operation of the propulsion units, the propulsive coefficient of the combination is determined from the expression

$$\eta_i^{(a)} = \frac{\eta_1 N_1 + \eta_2 N_2}{N_1 + N_2} = \frac{(P_1 + P_2)\,v}{75\left(\frac{P_1 v}{75\eta_1} + \frac{P_2 v}{75\eta_2}\right)} = \frac{\left(\frac{P_1}{P_2} + 1\right)\eta_1\eta_2}{\frac{P_1}{P_2}\eta_2 + \eta_1}, \qquad (1)$$

where

$$\eta_1 = \frac{2}{\sqrt{1 + \sigma_1} + 1}; \quad \sigma_1 = \frac{2P_1}{\varrho F_1 v^2}. \qquad (2)$$

The situation is analogous for η_2 and σ_2.

In addition

$$\frac{\sigma_1}{\sigma_2} = \frac{P_1}{P_2} \cdot \frac{F_2}{F_1}. \tag{3}$$

Let us use the notation

$$\frac{F_1}{F_1 + F_2} = \varphi; \quad \frac{F_2}{F_1 + F_2} = 1 - \varphi. \tag{4}$$

Then we get from equations (3) and (4) /129

$$\frac{P_1}{P_2} = \frac{\sigma_1}{\sigma_2} \cdot \frac{\varphi}{1 - \varphi}. \tag{5}$$

Substituting equations (2) and (5) in expression (1), we get the following expression for the propulsive coefficient of the first type of propulsion system

$$\eta_i^{(a)} = \frac{2\left(\frac{\sigma_1}{\sigma_2} \cdot \frac{\varphi}{1 - \varphi} + 1\right)}{\frac{\sigma_1}{\sigma_2} \cdot \frac{\varphi}{1 - \varphi}\left(1 + \sqrt{1 + \sigma_1}\right) + \left(1 + \sqrt{1 + \sigma_2}\right)}. \tag{6}$$

For the second modification,

$$\eta_i^{(б)} = \frac{2}{1 + \sqrt{1 + \sigma'}}$$

where

$$\sigma' = \frac{P_1 + P_2}{\frac{1}{2} \varrho (F_1 + F_2) v^2} = \sigma_1 \varphi + \sigma_2 (1 - \varphi). \tag{7}$$

where

$$\eta_i^{(5)} = \frac{2}{1+\sqrt{1+\sigma_1\varphi+\sigma_2(1+\varphi)}}. \tag{8}$$

In proceeding to analysis of the third modification, we must make the following stipulation. In the case of two active discs arranged in series and developing thrusts P_1 and P_2, the area of the second propulsion unit F^*_2 for a predetermined value of F_1 may not be arbitrarily assumed since it is necessary to satisfy the condition of a constant mass flowing through cross-sections F_1 and F^*_2. Then

$$F_2^* = \frac{2+a_1}{2+a_2}\cdot F_1, \tag{9}$$

where

$$a_1 = \frac{w_{a_1}}{v} = \sqrt{1+\sigma_1} - 1; \quad a_2 = \frac{w_{a_2}}{v} = \sigma_2\cdot\frac{1-\varphi}{\varphi}\cdot\frac{1}{1+\frac{1}{2}a_1} \tag{10}$$

(here and in the following discussion w_{aj} is the induced velocity of the j-th stage, w_a is the induced velocity of the propulsion unit as a whole).

For the case considered by V. M. Lavrent'yev [2], the propulsive coefficient is always lower than for a single-stage propulsion unit with identical parameters (i.e. the area F_1 and the thrust $P_1 + P_2$).

With a transition to the two-stage system, both propulsion units are ordinarily enclosed in the water guide channel. In this case, the area F_2 /130
may not satisfy condition (9), in particular, it may be taken as equal to F_1. The outlet cross-section for the flow channel in the given modification is defined by the condition

$$F_{out} = F_1\frac{1+a_1}{1+a}; \quad a = \frac{w_a}{v} = \frac{\sum_1^n w_{aj}}{v}. \tag{11}$$

The following formula may be used for the propulsive coefficient of an ideal two-stage system with a water flow channel:

$$\eta_{2st} = \frac{1}{1 + \frac{\sigma_e}{4} \cdot \frac{v}{v_{s_1}}} = \frac{1}{1 + \frac{\sigma_e}{4} \cdot \frac{1}{1 + a_1}}, \tag{12}$$

where

$$a = \frac{w_{a_1}}{v} = \frac{\sqrt{1 + 2\sigma_1} - 1}{2}. \tag{13}$$

The load factor for a two-stage propulsion unit may be expressed in terms of the load factors for each stage:

$$\sigma_{2st} = \frac{P_1 + P_2}{\frac{\varrho}{2} F_1 v^2} = \sigma_1 + \sigma_2 \frac{F_2}{F_1} = \sigma_1 + \sigma_2 \frac{1 - \varphi}{\varphi}. \tag{14}$$

Substituting equations (13) and (14) in formula (12), we get

$$\eta_{2st} = \frac{1}{1 + \frac{1}{2}\left(\sigma_1 + \sigma_2 \frac{1 - \varphi}{\varphi}\right) \frac{1}{\sqrt{1 + 2\sigma_1} + 1}}. \tag{15}$$

Equations (6), (8) and (15) may be used for comparing the efficiency of the given modifications of parallel and series arrangement of propulsion units for predetermined distribution of thrusts P_1 and P_2 and hydraulic cross-sections F_1 and F_2. For this purpose, curves of $\eta = f(\sigma_1)$ were plotted for a fixed value of $\sigma_2 = 0.2$ and three values of ϕ (Figure 2). According to Fig. 2, for identical initial data, the propulsive coefficient of the installation in the second modification is somewhat greater than the value of that for the installation made according to the first modification. Comparison of the efficiency of the first modification and the two-stage system is easily followed on Figure 3 where the values of $\eta_{2st}/\eta_i^{(a)} = F(\sigma 1/\sigma 2)$ are plotted for $\sigma_2 = 0.2$ and three values of ϕ. In the case of the two stage system,

$\eta_{2st}/\eta_i^{(a)}$ may be greater than or less than 1 depending upon the relationship between ϕ and σ_1/σ_2. Typically, the propulsive coefficient of an ideal two-stage system reaches higher values with an increase in the parameters ϕ and σ_1 than in the case of the first modification of the system, which agrees /132
satisfactorily with formula (16). The values of σ_2 and ϕ may be analogously analyzed.

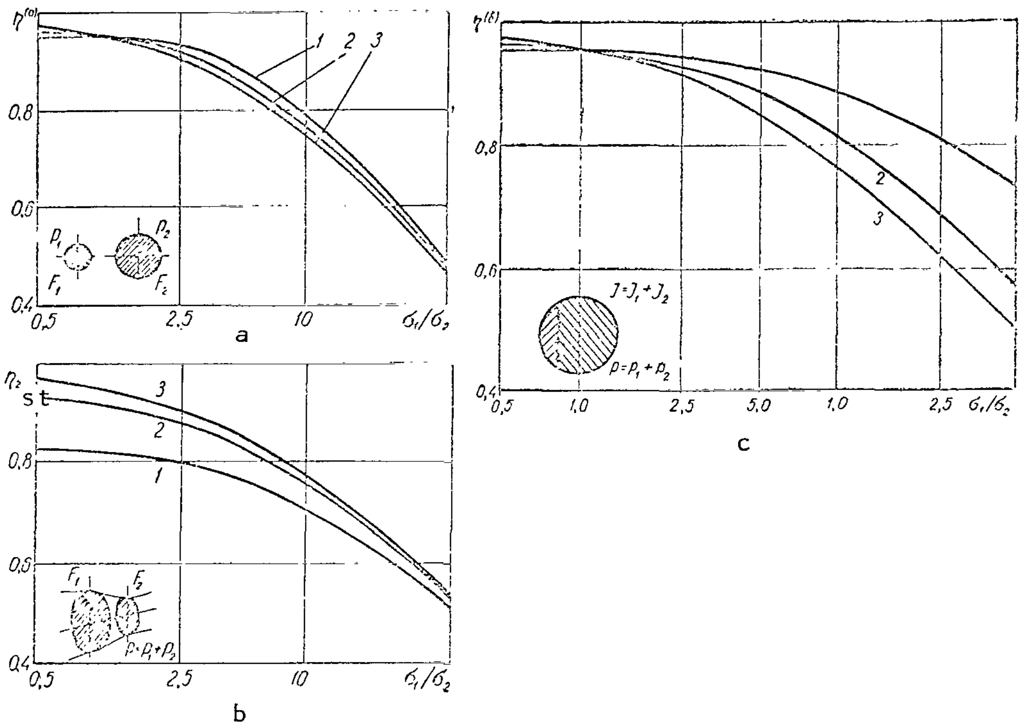

Fig. 2. Curves for Efficiency as a Function of the Ratio σ_1/σ_2 at Various Values of ϕ for the Three Modifications of Combining Propulsion Units: a, First Modification; b, Second Modification; c, Third modification (1, ϕ = 0.2, 2, ϕ = 0.5, 3, ϕ = 0.8).

If we remain within the framework of the theory of an ideal propulsion unit, in particular if we operate with the concept of an ideal fluid, we cannot evaluate the other factors which limit the arbitrary combination of parameters in a gas-hydrojet-screw system. The existing theory of an ideal cavitating screw [1] limits the over-all load factor at a given forward velocity when the continuity of the flow is interrupted at the inlet section of the screw (cavitation), which inevitably takes place in a real fluid.

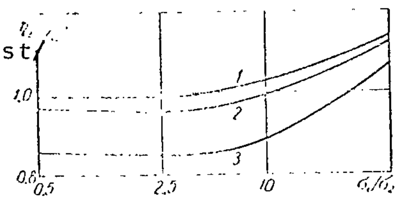

Fig. 3. Curves for the Efficiency Ratio of Various Combination Propulsion Systems as a Function of the Ratio σ_1/σ_2: 1, $\phi = 0.8$; 2, $\phi = 0.5$; 3, $\phi = 0.2$.

The limiting value σ_{cr} at which cavitation takes place is determined by the formula

$$\sigma_{cr} = 4\left[(1 + \varkappa) - \sqrt{1 + \varkappa}\right]. \tag{16}$$

The rate of flow beneath the screw which is reached in this case is

$$v_{s(cr)} = v\sqrt{1 + \varkappa}, \tag{17}$$

where $\varkappa = \dfrac{p_0 - p_d}{\frac{\rho}{2} v^2}$ is the cavitation number.

The necessity of accounting for this condition limits the possible values of the ratio P_1/P_g in a staged gas-hydrojet-screw installation where a permissible propulsive coefficient is maintained. Let us analyze the limiting values of P_1/P_g and the propulsive coefficient of the installation for various values of κ and other corresponding conditions.

The propulsive coefficient of an idealized gas-hydrojet may be represented in the form [2]

$$\eta_{gi} = \eta_g \eta_i = \frac{\ln\left(\delta \cdot \dfrac{1}{1 + \dfrac{\gamma H}{p_0}}\right)}{\dfrac{k}{k-1}\left(\delta^{\frac{k-1}{k}} - 1\right)} \cdot \frac{1}{1 + \dfrac{\sigma_m}{2}}, \tag{18}$$

where δ is the degree of air compression in the compressor, /133

$$\delta = \frac{p_k}{p_0} = f\left(\sigma_m; \frac{\gamma H}{p_0}; \varkappa\right);$$

σ_m is the load factor of the gas-hydrojet,

$$\sigma_m = P_g/mv = w_{a_2}/v;$$

H is the immersion of the screw axis; k is the adiabatic exponent.

Curves are given in [2] of the function $\eta_{g_i} = f(\sigma_m)$ for a number of values of κ at a fixed value of $\gamma H/P_0 = 0.2$. Using these curves, we can find a range of values of σ_m which are close to the optimum for a gas-hydrojet engine. Let us find the relationship between the quantities σ_{cr} and σ_m for a gas-hydrojet-screw installation of staged type.

Obviously, the total thrust of the staged propulsion unit is

$$P_e = P_1 + P_g = \rho F_1 v_{s_1} (w_{a_1} + w_{a_2}) \tag{19}$$

or

$$\sigma_e = \frac{P_e}{\frac{\rho}{2} F_1 v^2} = 2 \frac{v_{s_1}}{v} (\sqrt{1 + \sigma_1} - 1 + \sigma_m),$$

where F_1 and v_{s_1} are the inlet cross-section and the flow rate beneath the screw.

The maximum value of σ_e will be reached with fulfillment of condition (17):

$$\sigma_{e(cr)} = 2\sqrt{1+\varkappa}\,(\sqrt{1+\sigma_1} - 1 + \sigma_m). \tag{20}$$

From this expression, knowing κ and taking account of equation (20), we find $\sigma_1 = 2P_1/\rho F_1 v^2$ for a number of σ_m. Then

$$\frac{P_1}{P_g} = \frac{\sigma_1}{\sigma_m} \cdot \frac{v}{2v_{s_1}} = \frac{\sigma_1}{\sigma_m} \cdot \frac{1}{\sqrt{1+\varkappa}}$$

The maximum values of ratios P_1/P_g for a number of κ are given in Figure 4. As this figure shows, there is a considerable reduction in the fraction of the thrust attributable to the screw as the forward velocity of the vessel increases for a given value of σ_m. A further increase in the ratio P_1/P_g involves an appreciable reduction in the efficiency of the gas-hydrojet stage. The practicable values of the propulsive coefficient calculated from formula (1) for a gas-hydrojet-screw installation of the stage type are shown in Figure 5 for the conditions given above (the broken curve shows the propulsive coefficient for an ideal engine calculated from formula (16) for $\sigma_{e(cr)}$,). There is little change in the maximum practicable values of the efficiency in the given range of variation in parameters κ P_1/P_g and consequently $\sigma_{e(cr)}$. /134

This confirms the design feasibility of a staged gas-hydrojet-screw system within the framework of the given limitations (subcavitation conditions).

The problem of optimum design of a real installation reduces to analyzing a number of variations with regard to the favorable shape for the flow section, internal and blade losses and the maximum feasible values of σ_m and the ratio P_1/P_g.

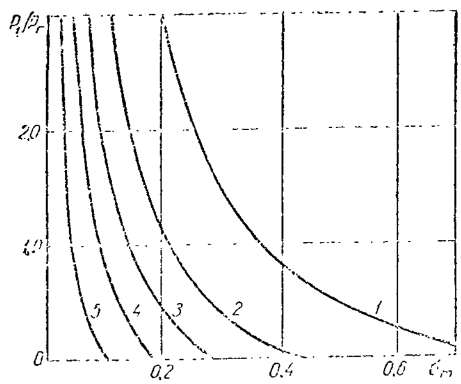

Fig. 4. Curves for P_1/P_g as a Function of σ_m at Various Values of κ: 1, $\kappa = 1.0$, $\sigma_{cr} = 2.3$; 2, $\kappa = 0.5$, $\sigma_{cr} = 1.02$; 3, $\kappa = 0.3$, $\sigma_{cr} = 0.64$; 4, $\kappa = 0.2$, $\sigma_{cr} = 0.4$; 5, $\kappa = 0.1$, $\sigma_{cr} = 0.2$.

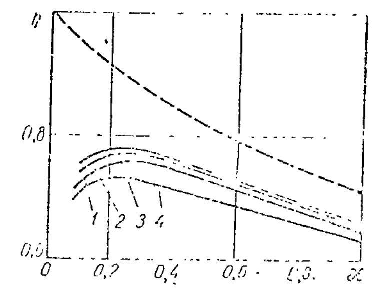

Fig. 5. Curves for Efficiency of Gas-Hydrojet-Screw Installations as a Function of κ: 1, $P_1/P_g = 2.0$; 2, $P_1/P_g = 1.5$; 3, $P_1/P_g = 1.0$; 4, $P_1/P_g = 0.5$.

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ON SELECTING THE TYPE OF ENGINE FOR HIGH SPEED SUBMARINES

V. A. Grigor'yev

ABSTRACT: The article contains a brief survey of non-Soviet research on water-steam rocket engines. Problems of the internal ballistics of the water-steam engine are formulated in the general case.

Extensive research has recently been undertaken abroad on reducing the drag and increasing the velocity of underwater vehicles, which involves solving the problem of selecting a type of engine. The screw type of propulsion unit is preferable for velocities of less than 90 knots. Coaxial screws and screws in nozzles have shown satisfactory results. The hydrojet is considered as the next step in the design of compact propulsion units. /135

Considerable attention is being given to the feasibility of using jet engines on submarines. Two types of jet engines are used: those which operate without using the surrounding water to produce thrust, and hydrojet engines which operate with the use of the surrounding water.

A great disadvantage of jet engines which are independent of the surrounding water is their extremely low thrust efficiency (η_{nd} = 0.05 - 0.1). Hydrojet engines have a higher thrust efficiency (η_{nd} = 0.7 - 0.8) as compared with those which are independent of the surrounding medium.

An interesting problem is the motion of underwater bodies in a cavity. Under conditions of completely developed cavitation, an underwater vehicle should move within a cavity which closes behind the vehicle; the vehicle would be controlled and stabilized either by foils which extend beyond the limits of the cavity or by jet rudders. Developed cavitation conditions could be produced either artificially or by reaching high velocities (v = 100 knots), which requires tremendous expenditures of energy. A jet engine is the only power plant which could be considered for motion of a vehicle within a stable cavity.

Ordinary water heated to the saturation state may be used to produce reactive thrust by escape of a jet. The operating principle of an engine of this type is as follows. A tank is filled with boiling water at saturation pressure P_1 and corresponding temperature t_1 or with cold water which is then heated. The boiling water escapes through a nozzle under saturation pressure. The drop in pressure during escape of the water causes violent formation of steam in the jet nozzle. In this case there is a sharp increase in the volume of the steam, and the hot water escaping through the nozzle takes on properties /136 similar to those of an expanding gas. The water loses heat of vaporization r, and thus the temperature of the water drops together with the pressure so that the water is cooled at the nozzle outlet to the final state P_2, t_2. During escape of the water, the pressure in the tank also drops which causes partial leveling of the water and a corresponding reduction in temperature. Several types of hot-water rockets have been patented in West Germany, but they have not been used because of imperfections. [1] gives the history of development of water-steam rockets with a theoretical investigation of the problem of using water rockets as boosters during vertical takeoff.

The water-steam engine has a comparatively low specific thrust (40-60 kg·product sec/kg), but this advantage is compensated by the low cost of a unit of thrust. It should be pointed out that studies of water-steam engines are in the engineering stage. Problems of the hydrodynamics of escape of the water-steam mixture, reduction in the specific thrust with a pressure drop in the tank, the inclination of boiling water to delay the boiling process during expansion in the nozzle and in the case of a phase displacement, etc. have not been sufficiently studied since they have been chiefly solved experimentally. The drop in pressure in the tank as the water escapes $P_1(\tau)$ and the reduction in temperature $t_1(\tau)$ cause a drop in thrust (Fig. 1). It has been proposed that the water be heated under pressure to balance the specific thrust of the engine. The Fairchild Company (United States) has proposed burning a solid fuel inside the tank to compensate for the heat of the steam formation [2]. Since there have been no data on two-phase discharge, heat transfer during passage of gases through the water and heating, the operating conditions of the engine have been experimentally determined. The effect of condensation discontinuity on its position and the efficiency of the nozzle have remained unexplained.

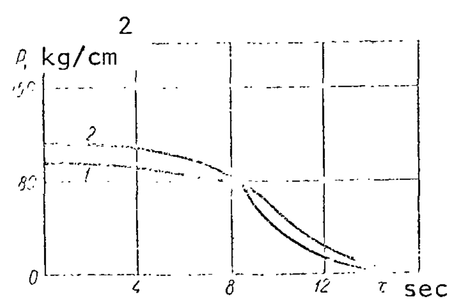

Fig. 1.

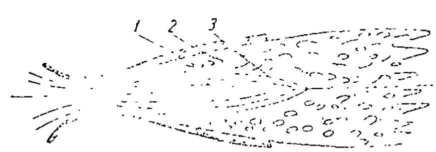

Fig. 2.

The most complex problem is determining the state of the moving steam-water mixture. The process of escape of the heated water is characterized by the fundamental equations of hydrodynamics and energy flow: equations of motion, equation of continuity, equation of energy and equation of state.

Basically, there is no difference between the molecular nature of adiabatic steam formation and the supply of heat through a wall. The difference is that the adiabatic steam formation process is due to the internal heat of the fluid liberated with the drop in pressure. In long nozzles, the steam /137 may be formed with simultaneous nearly thermodynamic equilibrium of the steam-liquid mixture. The degree of completion of the process of steam formation during escape of boiling water may be characterized by the ratio of the temperature at the outlet to the boiling temperature at the final pressure t_2/t_b.

The vapor formed on the surface of the jet, as it moves along the walls of the nozzle, leads to detachment of the jet from the wall. As a result, the cross-section of the liquid decreases with respect to the length of the nozzle. The flow in the nozzle is divided into a liquid film on the wall (1), a vapor ring (2) and a core of heated water (3) (Figure 2).

As the heated water escapes, it is necessary to account for the non-uniformity in phase distribution with respect to the cross-section of the nozzle, the difference in relative velocities of film, vapor and water core, the change in steam concentration x, the pressure gradient with respect to the length of the nozzle dP/dl, the adiabatic exponent $k(T,x)$, the Reynolds number, the area F_f/F occupied by the film, the slip coefficient of the phases $\nu = C_l/C_f$ and the phase temperature difference ΔT. Thus in order to close the given system of equations, it is necessary to add an equation for the change in volumetric concentration of the phases, an equation of phase velocity and an equation of heat interaction between phases.

As has been pointed out, the tank is filled with hot water or with water which has been heated. Even in the most successful experiments, two hours were required for starting the engine. The use of hydrojet fuel (lithium, potassium, sodium, etc.) has made it possible to do away with the boiler and heater. Work along these lines is being done by the Aerojet General Company in the United States. Successful research with hydrojet fuels has been carried out in Italy [3,4]. The chemical reagent produced by reaction with the /138 water in the rocket tank raises its temperature to 300°C in 0.3 sec. The use of hydrojet fuels not only eliminates the need for boilers and heaters, but raises the specific thrust to 360 dkN·sec/dkN. Investigations of the theoretical fundamentals of the internal ballistics of hydrochemical jet engines have not been completed. Particular attention is being given to experiments and searches for effective high-calorie hydrojet fuels.

By using the equations of internal ballistics--the equation of energy transformation, the equation of hydrojet fuel combustion, and the equation of motion of the vehicle--we may explain the relationships between the components of internal ballistics (pressure P, temperature t, volume V, coefficient of mass mixture m_1/m_2 in the mixing chamber, heat-transfer coefficient, etc.) and external ballistics (velocity v, length of the path L, time of motion τ) of the vehicle as a whole.

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ON EVALUATING THE TECHNICAL AND ECONOMIC INDICES OF ROCKET-HYDRAULIC TYPE HYDROJET ENGINES

V. A. Grigor'yev

ABSTRACT: The calculated characteristics of various fuels are compared. It is noted that the use of metal fuels has made it possible to increase the specific thrust of hydrojet engines by 30-40%. At the same time, the specific fuel consumption for a hydrojet engine is 40-50% less than that for a ramjet engine.

The use of direct reaction engines makes it possible to increase the velocity of a vehicle in media with different densities (water, air). In analyzing the power plant of the heat engine and propulsion unit, principal attention should be given to evaluating this power plant as a propulsion unit. /139

The indices of the power plant are determined in the absence of drag; discharge of the combustion products is characterized by constant velocity v_∞ and constant thrust P. The principal criteria in comparing jet engines are the specific thrust p, the over-all efficiency η_0 and the specific fuel consumption C_{sp}.

Table 1 summarizes the results of comparative calculations which may be used for first-approximation evaluation of p for several types of fuels under conditions of an ideal cycle (total combustion, absence of dissociation, the process of expansion and exhaust conforming to isentropic law). By using metal as a fuel[1], the specific thrust may be increased by 30-40% as compared with liquid fuels.

The thermal effect of the reaction of metal oxidation determined by Hess law is equal to the sum of the heats of formation of the final products of the reaction minus the sum of the heats of formation of the initial substances:

$$[\Delta H_{form}(nR) + \Delta H_{form}(xn + y)] - \{\Delta H_{form} nR\,(OH)_x] + \\ + \Delta H_{form}(H_2) + \Delta H_{form}[(y - 1)\,H_2O]\} = 0, \qquad (1)$$

[1]The idea of using metal fuels in jet engines was first proposed by F. A. Tsander in a lecture given in December 1923 at a meeting of a section of the Moscow Society of Astronomy Amateurs (*"Tekhnika i zhizn'", No. 13, 1924*).

Table. Properties of Various Types of Fuel

Type of Fuel	Chemical Symbol	Density, g/cm	Molecular Weight	Melting Point, °C	Boiling Point, °C	Reaction Equation	Reaction Products: Molecular Weight	Reaction Products: Qty of Molecular Hydrogen Liberated, l/g of substnc.	Reaction heat per unit wt. kcal/kg	Reaction heat per unit vol. kcal/l	Theoretical Exhaust Velo., m/sec	Theoretical specific Thrst per unit wt. dkN·sec/dkN	Theoretical Specific Thrst per unit vol. dkN/sec·l
Solid Fuel	—	—	—	—	—	—	—	—	1150	—	3105	316	—
Ethyl Alcohol plus Oxygen	C_3H_5OH	0,789	—	—	78,3	$C_2H_5OH+3O_2=2CO_2+3H_2O$	142,1	—	2070	—	4164	424	—
Aluminum	Al	2,7	26,98	658	2500	$2Al+\frac{3}{2}O_2=Al_2O_3$	101,94	—	3730	—	5590	570	—
Aluminum	Al	2,7	26,98	657	2500	$Al+3H_2O=Al(OH)_3+\frac{3}{2}H_2$	81,03	1,25	4150	—	5900	601	—
Zirconium	Zr	6,4	91,22	1852		$Zr+4H_2O=Zr(OH)_4+2H_2$	163,3	0,492	2710	—	4760	486	—
Magnesium	Mg	1,74	24,32	657	1102	$Mg+\frac{1}{2}O_2=MgO$	40,32	—	3623	—	5510	562	—
Magnesium	Mg	1,74	24,32	657	1102	$Mg+2H_2O=Mg(OH)_2+H_2$	60,32	0,924	3800	5000	5640	575	660
Lithium	Li	0,534	6,94	179	1372	$2Li+\frac{1}{2}O_2=Li_2O$	29,88		4764		6320	644	
Lithium	Li	0,534	6,94	179	1372	$2Li+2H_2O=2LiOH+H_2$	49,88	1,62	5340		6700	683	
Sodium	Na	0,97	22,99	99,7	883	$2Na+\frac{1}{2}O_2=Na_2O$	60,0		1650		3718	379	
Sodium	Na	0,97	22,99	99,7	883	$2Na+2H_2O=2NaOH+H_2$	81,98	0,49	1095		3030	309	

where ΔH_{form} is the heat of formation of the substance; x is the valence of the metal; y is the number of excess moles of water; n = 2 corresponds to an odd metal valence; n = 1 corresponds to an even metal valence.

Let us note that dissociation is observed with retardation of the chemical reaction when $T \geq 2{,}000°K$. As the flow expands, dissociation is followed by recombination, association of the molecules, i.e. the appearance of paired structured molecules, and then condensation. All this distinguishes the real process from the ideal process. Losses in the real cycle are characterized by the internal efficiency: /141

$$\eta_{in} = \eta_{th}\eta_{reac}\eta_{mech}\eta_{mix} \ . \tag{2}$$

In this paper, only the thermal efficiency η_{th} is taken into account in calculating the ideal engine.

The heat liberated after the metal fuel mass m_1 is reacted with the oxidizer (intake water) mass m_2 may be transferred to an additional mass of intake water m_3, which raises the thermal efficiency of the cycle. The increase in η_{th} is explained by the increase in the specific vapor concentration in the reaction products, the reduction in molecular heat μc_p, and the consequent reduction in the adiabatic exponent $k = \mu c_p/\mu c_v$.

The results of calculations of the parameters of jet engines which use products of the reaction $H_g + yH_2O$ as the working medium are shown in Figures 1 and 2 for various values of the coefficient of excess intake water $k_2 = m_3/m_2$.

The hydrojet thrust for the theoretical operating state of a nozzle with regard to the stoichiometric coefficient $k_1 = m_2/m_1$ will be

$$P = m_1[(1 + k_1 + k_1k_2)(v_\infty - v_0) + v_0]. \tag{3}$$

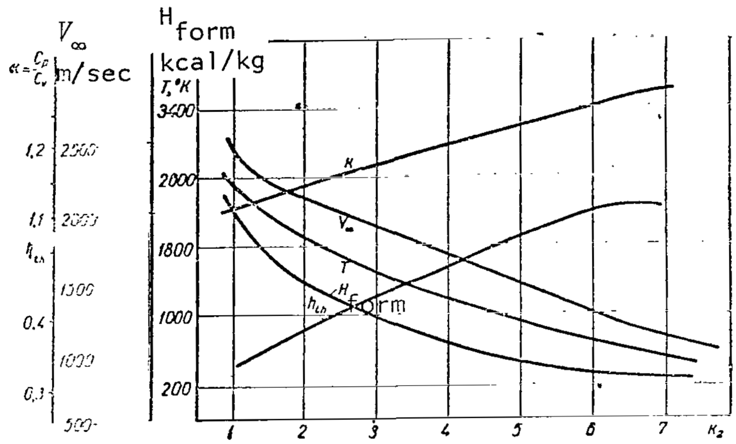

Fig. 1. Curves for the Thermodynamic Characteristics of a Hydrojet Engine as a Function of the Coefficient of Excess Intake Water k_2 (50:1).

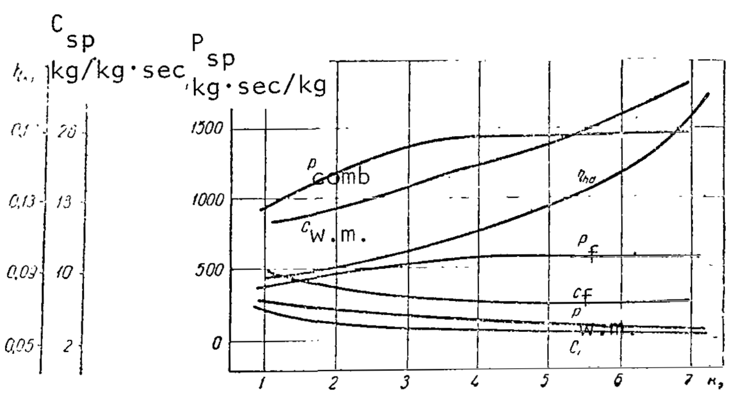

Fig. 2. Curves for Specific Thrust, Specific Fuel Consumption and Efficiency of a Hydrojet Engine as Functions of the Coefficient of Excess Intake Water k_2 (50:1).

In analyzing a hydrojet engine, it is preferable to consider not only the specific thrust of the engine, i.e. the ratio of the thrust to the consumption of working mass per second p_{wm}, but also the ratio of the thrust to the combustible mass p_{comb}, and to the fuel mass p_f: /142

$$P_{comb} = \frac{1}{g}\left[(1 + k_1 + k_1 k_2)(v_\infty - v_0) + v_0\right]; \tag{4}$$

$$p_f = \frac{1}{g(1 + k_1)} [(1 + k_1 + k_1 k_2)(v_\infty - v_0) + v_0]; \tag{5}$$

$$p_{w.m.} = \frac{1}{g(1 + k_1 + k_1 k_2)} [(1 + k_1 + k_1 k_2)(v_\infty - v_0) + v_0]. \tag{6}$$

Thus as the additional mass of intake water increases, the combustible mass p_{comb} increases while the specific thrust of the engine $p_{w.m.}$ decreases. The specific fuel consumption c_f is 2.5 kg/kg·sec at large values of k_2, i.e. this type of engine is economically superior to a ramjet.

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ON CALCULATING PRESSURE ENERGY DIFFUSION IN A FLOW WITH SHEAR

Ye. V. Yeremenko

ABSTRACT: An approximating expression is found for calculating pressure energy diffusion with regard to the finiteness of space.

The pulsation energy balance equation is utilized in the semi-empirical theory of turbulence proposed by A. N. Kolmogorov [3] which was developed and used by other authors for calculating flows with shear. The diffusion term of this equation /143

$$\sum_{m=1}^{3}\frac{\partial}{\partial x_m}\left[-\varrho\nu\frac{\partial E}{\partial x_m}+\left\langle u_m\left(\varrho\sum_{i=1}^{3}\frac{u_i^2}{2}+p\right)\right\rangle\right]$$

includes transfer of kinetic energy of turbulence E under the effect of viscosity (first term of the expression), turbulent diffusion of kinetic energy (second term of the expression) and pressure energy diffusion (third term of the expression).

Various methods are used by authors in their calculations to account for the diffusion term. In examining the ground layer of the atmosphere, A.S.Monin [4] considered only the second term--turbulent diffusion of kinetic energy--since the remaining terms were vanishingly small. In G. S. Glushko's paper [1] dealing with calculation of the boundary layer, the second and third terms of the expression are considered jointly and approximated in the form

$$\left\langle u_3\left(\varrho\sum_{i=1}^{3}\frac{u_i^2}{2}+p\right)\right\rangle\sim\nu'\frac{\partial E}{\partial x_3},$$

where ν' is a coefficient of the turbulent viscosity type.

This type of approximation for the diffusion of kinetic energy and the diffusion of pressure energy is frequently used [8]. However, A. A. Townsend [6] noted that although both these terms are important for energy transfer from one section of the flow to another, the second term of the /144 expression is local while the third is determined by the entire flow as a whole, and there is not necessarily any direct relationship between them.

Thus, there are sufficient grounds for considering these terms independently of one another. Pottal [10] isolated terms associated with pulsation pressure in recording the energy valence by components, and considered the correlation between pressure and velocity for an infinite region. According to experimental data given by Laufer [8], pressure energy diffusion in the greater part of the flow adjacent to the wall has an order identical to that of kinetic energy diffusion, where the signs of the corresponding terms are opposite to one another. This shows the significance of pressure energy diffusion and the difference between this type of diffusion and that of the kinetic energy of turbulence. We may assume that the derivation of an approximating expression for pressure energy diffusion will make it possible to improve the accuracy of computations in semi-empirical theory.

Applying the operation of divergence to Navier-Stokes equations, the pulsation pressure in an incompressible fluid may be written [10] in the form

$$\frac{1}{\varrho}\Delta p(r, r', t) = -\sum_{k=1}^{3}\sum_{i=1}^{3} 2\frac{\partial U_i(r+r', t)}{\partial x_k}\frac{\partial u_k(r+r', t)}{\partial x_i} - \sum_{k=1}^{3}\sum_{i=1}^{3}\left[\frac{\partial^2 u_k u_i}{\partial x_k \partial x_i}(r+r', t) - \frac{\partial^2 \langle u_k u_i\rangle}{\partial x_k \partial x_i}(r+r', t)\right]\ldots, \tag{1}$$

where U, u are the average and the pulsation velocities, respectively; p is the instantaneous value of the pulsation pressure.

Expression (1) is a Poisson equation which may be presented in a more compact form:

$$\frac{1}{\varrho}\Delta p(r, r', t) = -f(r, r', t). \tag{2}$$

The solution of equation (2) for infinite space with the use of the condition at infinity p = 0 gives [7]

$$\frac{p(r, t)}{\varrho} = \int_W \frac{f(r, r', t)}{r'} dW, \tag{3}$$

where r is the radius vector of the given point in space;

$$r' = \sqrt{\sum_{i=1}^{3} \xi_i^2} = \sqrt{(x_1 - x_{10})^2 + (x_2 - x_{20})^2 + (x_3 - x_{30})^2};$$

$$dW = d\xi_1 d\xi_2 d\xi_3.$$

We represent the derivative $\partial U_i/\partial x_k$ appearing in formula (1), as the Taylor series /145

$$\frac{\partial U_i}{\partial x_k}(r + r', t) = \frac{\partial U_i}{\partial x_k}(r, t) + \sum_{s=1}^{\infty} \frac{1}{s!} \sum_{n_1=1}^{3} \sum_{n_2=1}^{3} \cdots$$

$$\cdots \sum_{n_s=1}^{3} \frac{\partial^{s+1} U_i(r, t)}{\partial x_k \partial x_{n_1} \partial x_{n_2} \cdots \partial x_{n_s}} \xi_{n_1} \xi_{n_2} \cdots \xi_{n_s}.$$

We substitute the value of the function f(r, r', t) in expression (3). We multiply both members of equation (3) by $u_m(r, t)$ and differentiate with respect to x_m. Then taking the statistical average of the result and carrying out summation with respect to m, we get an expression for pressure energy diffusion in the form of a series:

$$\sum_{m=1}^{3} \frac{1}{\varrho} \frac{\partial \langle p u_m(r, t) \rangle}{\partial x_m} = \sum_m \sum_k \sum_i \frac{\partial^2 U_i}{\partial x_m \partial x_k} A_i^{km} +$$

$$+ \sum_m \sum_k \sum_i \sum_{n_1} \frac{\partial^3 U_i}{\partial x_m \partial x_{n_1} \partial x_k} {}_{n_1}A_i^{km} + \tag{4}$$

$$+ \sum_m \sum_k \sum_i \sum_{n_1} \sum_{n_2} \frac{\partial^4 U_i}{\partial x_m \partial x_{n_1} \partial x_{n_2} \partial x_k} {}_{n_2 n_1}A_i^{km} + \cdots.$$

In this equation

$$A_i^{km} = \frac{1}{2\pi} \int_W \frac{\partial R_k^m}{\partial x_i} \frac{dW}{r'}; \qquad {}_{n_1}A_i^{km} = \frac{1}{2\pi} \int_W \frac{\partial R_k^m}{\partial x_i} \xi_{n_1} \frac{dW}{r'};$$

$$ {}_{n_2 n_1}A_i^{km} = \frac{1}{2\pi} \int_W \frac{\partial R_k^m}{\partial x_i} \xi_{n_2} \xi_{n_1} \frac{dW}{r'}, \tag{5}$$

where the correlation function

$$R_k^m = \langle u_k(r + r', t)\, u_m(r, t)\rangle.$$

However, consideration of an infinite region does not fully correspond to the real conditions of motion of a flow with shear. The formal result of accounting for the finiteness of space in formula (5) is to change the limits of integration and the Green's functions.

In actuality, the solution of equation (2) for a half-space may be written in the form [9]

$$\frac{p}{\varrho}(r, t) = -\frac{1}{4\pi}\int\limits_{W}{}_{x_3>0} f(r, r', t)\left(\frac{1}{r'} + \frac{1}{r_1'}\right) dW, \tag{6}$$

where

$$r_1' = \sqrt{(x_1 - x_{10})^2 + (x_2 - x_{20})^2 + (x_3 + x_{30})^2}$$ /146

The boundary condition $(dp = 0)/(\partial x_3/x_3 = 0)$ is used in derivation of equation (6). Equation (2) may be solved not only for a region bounded by a single impermeable surface, but for two such surfaces (pressurized motion of a plane/parallel flow) or for a free surface. The boundary condition on a free surface is $p = 0$ $(x_3 = h)$. Let us select points outside the region which will be the mirror reflection of the given point (x_{10}, x_{20}, x_{30}) with respect to an impermeable bottom and the free surface. With multiple reflection of these points and application of the Green's formula

$$\begin{aligned} p(r, t) = &-\frac{1}{4\pi}\int\limits_{W}{}_{0<x_3<h} \Delta p \frac{\partial W}{r'} + \frac{1}{4\pi}\iint\limits_{S_1}{}_{x_3=0}\left(p\frac{\partial \frac{1}{r'}}{\partial x_3} - \frac{1}{r'}\frac{\partial p}{\partial x_3}\right) dS_1 + \\ &+ \frac{1}{4\pi}\iint\limits_{S_2}{}_{x_3=h}\left(p\frac{\partial \frac{1}{r'}}{\partial x_3} - \frac{1}{r'}\frac{\partial p}{\partial x_3}\right) dS_2 \ldots \end{aligned} \tag{7}$$

to the given region (for points lying outside the given region, the left-hand member of equation (7) is equal to zero), we add and subtract equation (7) with expressions for the points outside the region with regard to the boundary conditions

$$\left.\frac{\partial p}{\partial x_3}\right|_{x_3=0} = 0; \quad \left. p\right|_{x_3=h} = 0$$

to give

$$\frac{p}{\varrho}(r, t) = \frac{1}{4\pi} \int\limits_{W}{}_{0<x_3<h} f(r, r', t)\, G\, dW, \tag{8}$$

where the Green's function

$$G = \sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{\sqrt{(x_1 - x_{10})^2 + (x_2 - x_{20})^2 + [x_3 - (x_{30} + 2nh)]^2}} + \sum_{j=-\infty}^{+\infty} \frac{(-1)^j}{\sqrt{(x_1 - x_{10})^2 + (x_2 - x_{20})^2 + [x_3 - (-x_{30} + 2jh)]^2}}. \tag{9}$$

Hence the general form of expression (4) will not change when the finiteness of the region is taken into account, although limitation of the region of integration in equations (5) modifies these formulas to give /147

$$A_i^{km} = \frac{1}{2\pi} \int\limits_{W}{}_{0<x_3<h} \frac{\partial R_k^m}{\partial x_i} G\, dW; \quad {}_{n_1}A_i^{km} = \frac{1}{2\pi} \int\limits_{W}{}_{0<x_3<h} \frac{\partial R_k^m}{\partial x_i} \xi_{n_1} G\, dW;$$

$$ {}_{n_1 n_2}A_i^{km} = \frac{1}{2\pi} \int\limits_{W}{}_{0<x_3<h} \frac{\partial R_k^m}{\partial x_i} \xi_{n_1} \xi_{n_2} G\, dW \tag{10}$$

Since the form of the correlation function is not known beforehand, we assume that the turbulent characteristics may be expressed in terms of E and some integral scale L. With regard to expression (10), we find that A should be a function of E, L and h.

Let us consider only plain-parallel flows in the case of smoothly changing motion where

$$\sum_m \frac{1}{\varrho}\frac{\partial\langle p(r,t)\,u_m(r,t)\rangle}{\partial x_m} = \frac{\partial^2 U_1}{\partial x_3^2}A_1^{33} + \frac{\partial^3 U_1}{\partial x_3^3}{}_3A_1^{33} + \frac{\partial^4 U_1}{\partial x_3^4}{}_{33}A_1^{33}. \tag{11}$$

Let us limit ourselves to the first term of this series and use considerations of dimensionality for approximation. Taking account of the coefficient $H_{(R_E)}$ introduced by G. S. Glushko [1] for correcting the values of the coefficient of turbulent viscosity $\nu_t = \alpha H_{(RE)} L\sqrt{E}$ in the region near the wall and taking A_1^{33} as proportional to $E(a'h-b'L)$, which guarantees convergence of the series with regard to most of the terms, we get

$$\frac{1}{\varrho}\frac{\partial\langle p(r,t)\,u_3(r,t)\rangle}{\partial x_3} = aEh\left(b - H_{(R_E)}\frac{L}{h}\right)\frac{\partial^2 U_1}{\partial x_3^2}. \tag{12}$$

In deriving equation (12), a two-layer flow system was considered and I. K. Nikitin's data [5] were used for calculating

$$L = \frac{U_*^2\left(1 - \frac{x_3}{h}\right) - \nu\frac{dU_1}{dx_3}}{\sqrt{E}\frac{dU_1}{dx_3}} \quad (U_*$$

where U_* is dynamic velocity. The expression for determining the value of E is given in [2]. The extent to which equation (12) corresponds to Laufer's experimental data is shown in the Table.

In expression (12), the coefficients a = 0.13, b = 0.0073, and the coefficient H(Re) which depends on the Reynolds number for turbulence $Re = L\sqrt{E/\nu}$ was calculated from formulas given in [1] where $Re_0 = 22.0$.

TABLE

$x_3 = \frac{U_* x_3}{\nu}$	$E, \frac{cm^2}{sec^2}$	$\frac{\partial^2 U_1}{\partial x_3^2}, \frac{1}{cm \cdot sec}$	$-\frac{1}{\rho} \frac{d \langle Pu_3 \rangle}{\partial x_3}, \frac{cm^2}{sec^3}$		Discrepancy, %
			Theoretical Data	Experimental Data	
10	546	--9960.5	61800	59200	+4.4
20	713	—1474.2	10380	10400	—0.2
30	674	—575.5	2850	2660	+0.7
40	636	—297.5	0	0	—
60	499	—87.8	—290	—280	+3.5

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